THE PRINCIPLES

OF

MECHANICS:

DESIGNED FOR THE USE OF STUDENTS
IN THE UNIVERSITY.

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THE PRINCIPLES

OF

THE term *Mechanics* has at different times, and by different writers, been applied to branches of science essentially distinct from each other. It was originally confined to the doctrine of equilibrium, or the investigation of the proportion of powers when they balance each other.

Later writers, adapting the term to their discoveries, have used it to denote that science which treats of the nature, production, and alteration of motion; giving to the former branch, by way of contradistinction, the name of Statics.

Others, giving the term a still more comprehensive meaning, have applied it to both these sciences.

None of these definitions will exactly suit our present purpose; the first being too contracted; and the others much too extensive, for a treatise which is intended to be an introduction only, to the higher

branches of philosophy. Our system of Mechanics will comprise the doctrine of equilibrium, and so much of the science of motion as is necessary to explain the effects of impact and gravity.

SECTION I.

ON MATTER AND MOTION.

DEFINITIONS.

ART. (1.) MATTER is a substance, the object of our senses, in which are always united the following properties; *extension*, *figure*, *solidity*, *mobility*, *divisibility*, *gravity*, and *inactivity*.

(2.) *Extension* may be considered in three points of view: 1st. As simply denoting the part of space which lies between two points, in which case it is called *distance*. 2d. As implying both length and breadth, when it is denominated *surface* or *area*. 3d. As comprising three dimensions, length, breadth, and thickness; in which case it may be called *bulk*, *capacity*, or *content*. It is used in the last of these senses when it is said to be a property of matter.

(3.) *Figure* is the boundary of extension. The portions of matter, from which we receive our ideas of this substance, are bounded; that is, they have figure.

(4.) *Solidity* is that property of matter by which it fills space; or, by which any portion of matter

excludes every other portion from that part of space which it occupies; and thus it is capable of resistance and protrusion. "There is no idea which we receive more constantly from sensation than *solidity*. Whether we move or rest, in what posture soever we are, we always feel something under us that supports us, and hinders our farther-sinking downwards; and the bodies which we daily handle make us perceive that, whilst they remain between them, they do by an insurmountable force hinder the approach of the parts of our hands that press them *."

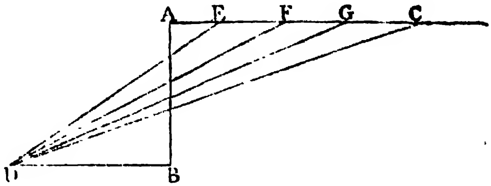
(5.) *Mobility*, or a capacity of being transferred from one place to another, is a quality found to belong to all bodies upon which we can make suitable experiments; and hence we conclude that it belongs to all matter.

(6.) *Divisibility* signifies a capacity of being separated into parts. That matter is thus divisible, our daily experience assures us. How far the division can actually be carried, is not so easily seen. We know that many bodies may be reduced to a very fine powder by trituration; by chemical solution, the parts of a body may be so far divided as not to be sensible to the sight; and by the help of the microscope we discover myriads of organized bodies, totally unknown before such instruments were invented. These and similar considerations, lead us to conclude, that the division of matter is carried to a degree of minuteness far exceeding the bounds of our faculties; and it seems not unreasonable to suppose, that this capacity of division is without limit; especially, as we can

* Locke's Essay, B. II. Ch. IV.

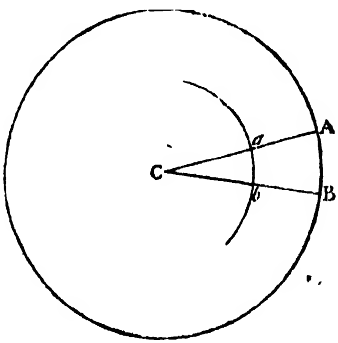
prove, theoretically, that any portion of extension is divisible into parts less and less without end*.

From the extremities of the line *AB*, draw *AC*, *BD*, parallel to each other, and in opposite directions; in *AC* take any number of points *E*, *F*, *G*, &c. and



join *DE*, *DF*, *DG*, &c. these lines will cut *AB* in different points; and since, in the indefinite line *AC*, an unlimited number of points may be taken, the number of parts into which *AB* is divisible, is indefinite.

This property of extension may also be proved



ex absurdo. If possible, let *AB* be the least portion

* Porro corporum partes divisas et sibi mutuò contiguas ab invicem separari posse ex phænominis novimus, et partes indivisas in partes minores ratione distingui posse ex mathematicâ certum est.

of a circular arc; take C the center, join CA, CB, and with the center C, and any radius Ca, less than CA, describe a circle cutting CA and CB in the points a and b; then, because AB and ab are similar arcs, they are as their radii; therefore ab is less than AB; or a portion of extension less than the least possible has been found, which is absurd. Hence, any portion of extension is divisible into parts less and less, without ever coming to a limit.

It has been supposed by some writers that there are certain indivisible particles of matter, of the same figure and dimensions, by the different modifications of which different bodies are formed. As no arguments are adduced in favour of this hypothesis, and as experiment seems to lead us to a contrary conclusion, we cannot allow it a place amongst the principles upon which our theory is to rest.

(7.) *Gravity* is the tendency which all bodies have to the center of the earth.

We are convinced of the existence of this tendency by observing that, whenever a body is sustained, it's pressure is exerted in a direction perpendicular to the horizon; and that, when every impediment is removed, it always descends in that direction.

The *weight* of a body is it's tendency to the earth, compared with the like tendency of some other body, which is considered as a standard. Thus, if a body with a certain degree of gravity be called one pound, any other body which has the same degree of gravity,

est. Utrum verò partes illæ distinctæ et nondum divisæ per vires naturæ dividi et ab invicem separari possint, incertum est. NEWT. Princip. L. III. Reg. 3.

or which by it's gravity will produce the same effect, under the same circumstances, is also called a pound; and these two together, two pounds, &c.

Gravity is not an accidental property of matter arising from the figure or disposition of the parts of a body; for then, by changing it's shape, or altering the arrangement of the particles which compose it, the gravitation of the mass would be altered. But we find that no separation of the particles, no change of the structure, which human power can effect, produces any alteration in the weight.

As gravity is a property belonging to every particle of a body, independent of it's situation with respect to other particles, the gravity of the whole is the aggregate of the gravities of all it's parts. Thus, though the weight of the whole is not altered by any division, or new arrangement of the particles, yet every increase or diminution of their number produces a corresponding increase or diminution of the weight.

Our present subject does not lead us to consider gravitation in any other point of view than simply as a tendency in bodies to the center of the earth, or to attend to it's effects at any considerable distance from the surface; it may not, however, be improper to observe that the operation of this principle is much more extensive. Every portion of matter gravitates towards every other portion, in that part of the system of nature which falls under our observation. The gravitation indeed of small particles towards each other is not perceived, on account of the superior action of the earth *; yet it has been found, by the

* This attraction is not sufficient to explain the common experiment

accurate observations of Dr. MASKELYNE, in Scotland, that the attraction of a mountain is sufficient to draw the plumb-line sensibly from the perpendicular.

Sir I. NEWTON has discovered that the moon is retained in her orbit by the agency of a cause similar to that by which a body falls to the ground, differing from it only in degree, and this in consequence of the greater distance of the moon from the earth's center. The same author has demonstrated that the planets are retained in their respective orbits by a principle of the same kind; and later writers have shewn, that the minutest irregularities in their motions may be satisfactorily deduced from the known laws of it's operation.

(8.) *Inactivity* may be considered in two lights: 1st. As an *inability* in matter to change it's state of rest or uniform rectilinear motion: 2d. As that quality by which it *resists* any such change*. In this latter sense it is usually called the *force of inactivity*, the *inertia*, or the *vis inertiæ*.

The inactivity of matter, according to the former explanation, is laid down as a law of motion; the truth of which we shall endeavour to establish in the next section.

ment of two particles of the same kind, as quicksilver, &c. when placed upon a smooth horizontal plane, running together. If the effect were not produced by some power different from gravitation, a drop of oil would run, in the same manner, towards a drop of water, which is not found to be the case.

* That is, a change from rest to uniform rectilinear motion, or a change in its uniform rectilinear motion. The term is sometimes applied to the resistance which a body makes to the production, or alteration of motion, when this resistance acts at a mechanical advantage, or disadvantage.

That a body *resists* any change in it's state of rest, or uniform rectilinear motion, is known from constant experience. We cannot move the least particle of matter without some exertion; nor can we destroy any motion without perceiving some resistance*. Thus, we see, in general, that *inertia* is a property inherent in all bodies with which we are concerned; different indeed in different cases, but existing, in a greater or less degree, in all. The quantity we are not at present considering; the existence of the property, every one, from his own observation, will readily allow. We know indeed from experience†, that the inertia of a body is not altered by altering the arrangement of it's parts; but if one portion of matter be added to another, the inertia of the whole is increased; and if any part be removed, the inertia is lessened. This clearly shews that it exists, independently, in every particle, and that the whole inertia is the aggregate of all it's parts.

Hence it follows, from our notion of quantity, that if to a body with a certain quantity of inertia, another, which has an equal quantity, be added, the whole inertia will be doubled; and that by the repeated addition of equal quantities, the whole inertia will be increased in the same proportion with the number of parts.

These properties, which are always found to exist together in the same substance, have sometimes been

* It must be observed, that this resistance is distinct from and independent of gravity; because it is perceived where gravity produces no effect: as, when a wheel is turned round *it's axis*, or a body moved along an horizontal plane.

† See Art. 25.

said to be essential to matter. Whether they are, or are not *necessarily* united in the same substance, it is impossible to decide, nor does it concern us to inquire. The business of natural philosophy is not to find out what might have been the constitution of nature, but to examine what it is in fact: and to account for the phænomena, which fall under our observation, from those properties of matter which we know by experience that it possesses.

(9.) By the *quantity of matter* in a body, we understand the aggregate of it's particles, each of which has a certain degree of inertia. Or, in other words, if we suppose bodies made up of particles, each of which has the same inertia, the quantity of matter in one body, is to the quantity of matter in another, as the number of such particles in the former, to the number in the latter *.

When we consider bodies as made up of parts, and compare them in this respect, it becomes necessary to give a definite and precise description of those parts; otherwise our notion of the quantity will be vague and inaccurate. Now the only properties of matter which admit of exact comparison, and which depend upon the number, and not upon the arrangement of the particles, are weight and inertia; either of which

* Quantitas materiæ est mensura ejusdem orta ex illius densitate et magnitudine conjunctim. NEWT. Princip. Def. 1.

Ejusdem esse densitatis dico, quarum vires inertiæ sunt ut magnitudines. Lib. III. Prop. 6. Cor. 4.

Attendi enim oportet ad punctorum numerum, ex quibus corpus movendum est conflatum. Puncta vero ea inter se æqualia censeri debent, non quæ æque sunt parva, sed in quæ eadem potentia æquales exerit effectus. EUL. Mech. 139.

may properly be made use of as a measure of the *quantity* of matter; and since, at a given place, they are proportional to each other, as we shall shew hereafter (Art. 25), it is of little consequence which measure we adopt. The inertia has been fixed upon, because the gravity of a body, though invariable at the same place, is different at different distances from the center of the earth; whereas, the inertia is always, and under all circumstances, the same.

The *density* of a body is measured by the quantity of matter in a *given bulk*; and it is said to be *uniform* when equal quantities of matter are always contained in equal bulks.

(10.) By *motion* we understand the act of a body's changing place; and it is of two kinds, *absolute* and *relative*.

A body is said to be in *absolute motion* when it is actually transferred from one point in fixed space to another; and to be *relatively in motion*, when it's situation is changed with respect to the surrounding bodies.

These two kinds of motion evidently coincide when the bodies, to which the reference is made, happen to be fixed. In other cases, a body relatively in motion, or relatively at rest, may or may not be absolutely in motion. Thus, a spectator standing still on the shore, if his place be referred to a ship which sails by, is relatively in motion; and the several parts of the vessel are at rest, with respect to each other, though the whole is transferred from one part of space to another.

The motion of a body is *swifter* or *slower*, according as the space passed over, in a given time, is *greater* or *less*.

When a body always passes over equal parts of space in equal successive portions of time, it's motion is said to be *uniform*. When the successive portions of space, described in equal times, continually increase, the motion is said to be *accelerated*; and to be *retarded*, when those spaces continually decrease. Also the motion is said to be *uniformly* accelerated or retarded, when the increments or decrements of the spaces, described in equal successive portions of time, are always equal.

(11.) The *degree* of swiftness or slowness of a body's motion is called it's *velocity*, and it is measured by the space, *uniformly described*, in a *given* time.

The given time, taken as a standard, is usually one second; and the space described is measured in feet. Thus, when v represents a body's velocity, v is the number of feet which the body would uniformly describe in one second.

If a body's motion be accelerated or retarded, the velocity at any point is not measured by the space actually described in a given time, but by the space which would have been described in the given time, if the motion had continued uniform, from that point; or had, at that point, ceased to increase or decrease.

(12.) Cor. 1. If two bodies move uniformly on the same line, in *opposite* directions, their relative velocity is equal to the *sum* of their absolute velocities, since the space by which they uniformly approach to, or recede from, each other, in any time, is equal to the sum of

the spaces which they respectively describe in that time.

When the bodies move in the *same* direction, their relative velocity is equal to the *difference* of their absolute velocities.

(13,) Cor. 2. When a body moves with an uniform velocity, the space described is proportional to the time of it's motion.

Let the body describe the space a in the time 1; then since the motion is uniform, it will describe the space ta in the time t; that is, the space described is proportional to the time.

(14.) Cor. 3. When bodies have different uniform motions, the spaces described are proportional to the times and velocities jointly*.

Let V and v be the velocities of two bodies A and B; T and t the times of their motions; S and s the spaces described. Also let S' be the space described by B in the time T:

$$\begin{array}{rl} \text{Then } & S : S' :: V : v \text{ (Art. 11)}, \\ & \underline{S' : s :: T : t} \text{ (Art. 13)}. \\ \text{Comp.} & S : s :: TV : tv; \end{array}$$

that is, $S \propto TV$ (*Alg.* Art. 195).

Ex. Let the times be to each other as 6 : 5, and the velocities as 2 : 3; then $S : s :: 2 \times 6 : 3 \times 5 :: 4 : 5$.

* Since the times and velocities may, in each case, be represented by numbers, there is no impropriety in speaking of their products. The truth of this observation will be evident, if the proposition be expressed in different words. When the uniform velocities of two bodies are in the ratio of the numbers V and v, and the times of their motions in the ratio of the numbers T and t, the spaces described are in the ratio of the numbers TV and tv.

(15.) Cor. 4. Since $S \propto TV$, we have $V \propto \frac{S}{T}$, and $T \propto \frac{S}{V}$, (*Alg.* Art. 205).

Ex. 1. Let *A* move uniformly through 5 feet in 3″, and *B* through 9 feet in 7″; required the ratio of the velocities.

$$V : v :: \frac{5}{3} : \frac{9}{7} \quad 35 : 27.$$

Ex. 2. Let *A*'s velocity be to *B*'s velocity as 5 to 4; to compare the times in which they will describe 9 and 7 feet respectively.

$$T : t \quad \frac{9}{5} : \frac{7}{4} \quad 36 : 35.$$

(16.) Cor. 5. Since the areas of rectangles are in the ratio compounded of the ratios of their sides, if the bases of two rectangles represent the velocities of two motions, and altitudes the times, the areas will represent the spaces described.

(17.) The *quantity of motion*, or *momentum*, of a body, is measured by the velocity and quantity of matter jointly.

Thus, if the quantities of matter in two bodies be represented by 6 and 7, and their velocities by 9 and 8, the ratio of 6×9 to 7×8, or 27 to 28, is *called* the ratio of their *momenta*.

(18.) Cor. 1. If M be the momentum of a body, Q it's quantity of matter, and V it's velocity, then since $M \propto QV$, we have $Q \propto \frac{M}{V}$; and $V \propto \frac{M}{Q}$.

Ex. If the quantities of motion be as 6 to 5, and the velocities as 7 to 8, what is the ratio of the quantities of matter?

Since $Q \propto \frac{M}{V}$, we have $Q : q :: \frac{6}{7} : \frac{5}{8} :: 48 : 35$.

(19.) Cor. 2. If M be given, $Q \propto \frac{1}{V}$; and conversely if $Q \propto \frac{1}{V}$, M is invariable. (*Algebra*, Art. 206.)

(20.) Whatever changes, or tends to change, the state of rest or uniform rectilinear motion of a body, is called *force*.

Thus, impact, gravity, pressure, &c. are called forces.

When a force produces it's effect instantaneously, it is said to be *impulsive**. When it acts incessantly, it is called a *constant*, or *continued* force.

Constant forces are of two kinds, *uniform* and *variable*. A force is said to be *uniform*, when it always produces *equal* effects in equal successive portions of time; and *variable*, when the effects produced in equal times are *unequal*.

Forces, which are known to us only by their effects, must be compared by estimating those effects under the same circumstances. Thus, impulsive forces must be measured by the whole effects produced; uniform forces, by the effects produced in equal times; and variable forces, by the effects which would be produced

* Though we cannot conceive finite effects to be produced otherwise than by degrees, and consequently in successive portions of time; yet when these portions are so small as not to be distinguishable by our faculties, the effects may be said to be instantaneous.

in equal times, were the forces to become and continue uniform during those times.

The effects produced by the actions of forces are of two kinds, velocity and momentum; and thus we have two methods of comparing them, according as we conceive them to be the causes of velocity or momentum.

(21.) The *accelerating force* is measured by the *velocity* uniformly generated in a given time, no regard being had to the quantity of matter moved.

Thus, if the velocities uniformly generated, in two cases, in equal times, be as 6 to 7, the accelerating forces are *said* to be in that ratio.

The accelerating force of gravity, at the same place, is invariable; for all bodies falling freely, in an exhausted receiver, acquire equal velocities in any given time.

(22.) The *moving force* is measured by the *momentum* uniformly generated in a given time.

If the momenta thus generated, in two cases, be as 14 to 15, the moving forces are *said* to be in that ratio.

(23.) Cor. 1. Since the momentum is proportional to the velocity and quantity of matter, the moving force varies as the accelerating force and quantity of matter jointly.

The moving force of gravity varies as the quantity of matter moved, because the accelerating force is given (Art. 21).

(24.) Cor. 2. Hence it follows that the accelerating force varies as the moving force directly, and the quantity of matter inversely.

Prop. I.

(25.) *The vis inertiæ of a body is proportional to it's weight.*

The inertia, as was observed on a former occasion, is the resistance which a body makes to any change in it's state of rest or uniform rectilinear motion (Art. 8.); and this resistance is manifestly the same in two bodies, if the same force, applied in the same manner, and for the same time, communicate to each of them the same velocity.

Let two bodies, *A* and *B*, equal in *weight*, be placed in two similar and equal boxes, which are connected by a string passing over a fixed pulley; then these will exactly balance each other; and if the whole be put into motion, the gravity can neither accelerate nor retard that motion; the whole resistance therefore to the communication of motion in the system, arises from the inertia of the weights, the inertia of the string and pulley *, the friction upon the axis, and the resistance of the air †.

Now let a weight *C* be added on one side, and let the velocity generated in any given time, in the whole system, by this additional weight, be observed.

Then in the place of *A*, or *B*, substitute any other

* See Note, page 8.

† This experiment may be made with great accuracy by means of a machine, invented by Mr. Atwood, for the purpose of examining the motions of bodies when acted upon by constant forces. This machine is described in his well-known treatise on the *Rectilinear Motion and Rotation of Bodies*, (p. 299.)

mass of the same weight, and it will be found that *C* will, in the same time, generate the same velocity in this system as in the former; and, therefore, the whole resistance to the communication of motion must be the same. Also the inertia of the string and pulley, the friction of the axis, and the air's resistance, are the same in the two experiments; consequently, the resistance arising from the inertia of the weights is the same: That is, so long as the weight remains unaltered, whatever be the form or constitution of the body, the inertia is the same.

Also, since the whole quantity of inertia is the aggregate inertia of all the parts, if the *weight* be doubled, an equal quantity of *inertia* is added to the former quantity, or the whole *inertia* is doubled; and in the same manner, if the *weight* be increased in any proportion, by the repeated addition of equal *weights*, the *inertia* is increased in the same proportion.

It may be observed, that the velocity generated in a given time, is the same, whether the system begins to move from rest or not; therefore the inertia is the same, whether the system be at rest or in motion.

(26.) Cor. Since the quantity of matter is measured by the inertia (Art. 9.), it is also proportional to the weight.

SECTION II.

ON THE LAWS OF MOTION.

THE FIRST LAW.

(27.) *If a body be at rest, it will continue at rest, and if in motion, it will continue to move uniformly forward in a right line, till it is acted upon by some external force.*

That a body at rest cannot put itself in motion, we know from constant and universal experience.

That a body in motion will continue to move uniformly forward in a right line till it is acted upon by some external force, though equally certain, is not, it must be allowed, equally apparent; since all the motions which fall under our immediate observation, and rectilinear motions in particular, are soon destroyed. If however we can point out the causes which tend to destroy the motions of bodies, and shew, experimentally, that, by removing some of them and diminishing others, the motions continually become more uniform and rectilinear, we may justly conclude that any deviation from the first direction, and first velocity,

must be attributed to the agency of external causes; and that there is no tendency in matter itself, either to increase or diminish any motion impressed upon it.

Now the causes which retard a body's motion, besides collision, or the evident obstruction which it meets with from sensible masses of matter, are gravity, friction, and the resistance of the air; and it will appear, by the following experiments, that when these are removed, or due allowance is made for their known effects, we are necessarily led to infer the truth of the law above laid down.

1st. If a ball be thrown along a rough pavement, it's motion, on account of the many obstacles it meets with, will be very irregular, and soon cease; but if it be bowled upon a smooth bowling-green, it's motion will continue longer, and be more rectilinear; and if it be thrown along a smooth sheet of ice, it will preserve both it's direction and it's motion for a still longer time.

In these cases, the gravity, which acts in a direction perpendicular to the plane of the horizon, neither accelerates nor retards the motion; the causes which produce the latter effect are collision, friction, and the air's resistance; and in proportion as the two former of these are lessened, the motion becomes more nearly uniform and rectilinear.

2d. When a wheel is accurately constructed, and a rotatory motion about it's axis communicated to it, if the axis, and the grooves in which it rests, be well polished, the motion will continue a considerable time; if the axis be placed upon friction wheels, the motion will continue longer; and if the apparatus be

placed under the receiver of an air pump, and the air be exhausted, the motion will continue, without visible diminution, for a very long time.

In these instances, gravity, which acts aqually on opposite points of the wheel, neither accelerates nor retards the motion; and the more care we take to remove the friction, and the resistance of the air, the less is the first motion diminished in a given time.

3d. If a body be projected in any direction inclined to the horizon, it describes a curve, which is nearly the common parabola. This effect is produced by the joint action of gravity, and the motion of projection; and since the effect produced by the former is known, the effect produced by the latter may be determined. This, it is found, would carry the body uniformly forward in the line in which it was projected; as will fully appear when we come to the doctrine of projectiles. The deviation of the curve described from the parabolic form is sufficiently accounted for by the resistance of the air.

From these, and similar experiments, we are led to conclude that all bodies in motion would uniformly persevere in that motion, were they not prevented by external impediments; and that every increase or diminution of velocity, every deviation from the line of direction, is to be attributed to the agency of such causes.

(28.) It may not be improper to observe, that this law suggests two methods of distinguishing between absolute motions, and such as are only apparent; one, by considering the causes which produce the motions;

and the other, by attending to the effects with which the motions are accompanied *.

1st. We may sometimes distinguish absolute motion, or change of absolute motion, from that which is merely apparent, by considering the causes which produce them.

When two bodies are absolutely at rest, they are relatively so; and the appearance is the same, when they are moving in the same direction, at the same rate; a relative motion therefore can only arise from an absolute motion, or change of absolute motion, in one or both of the bodies. We have seen also, in the last article, that motion, or change of motion, cannot be produced but by force impressed; and therefore, if we know that such a cause exists, and acts upon one of the bodies, and not upon the other, we conclude that the relative motion arises from a change in the state of rest, or absolute motion of the former; and that with respect to the latter, the effect is merely apparent. Thus, when a person on shipboard observes the coast receding from him, he is convinced that the appearance arises from a motion, or change of motion, in the ship, upon which a cause, sufficient to produce this effect, acts, namely, the force of the wind or tide.

The precession of the equinoxes arises from a real motion in the earth, and not from any motion in the heavenly bodies; because we know that there is a force impressed upon the earth, which is sufficient to account for the appearance.

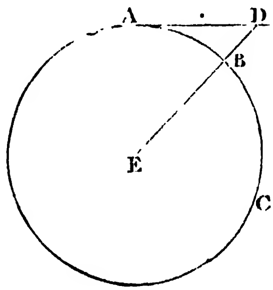
2d. Absolute motion may sometimes be disinguished from apparent motion, by the effects produced.

* Newt. Princip. Schol. ad Def.

If a body be absolutely in motion, it endeavours by it's inactivity to proceed in the line of it's direction; if the motion be only apparent, there is no such tendency.

It is in consequence of the tendency to persevere in rectilinear motion that a body revolving in a circle constantly endeavours to recede from the center. The effort thus produced is called a *centrifugal force;* and as it arises from absolute motion only, whenever it is observed, we are convinced that the motion is real.

In order to see the nature and origin of this force, suppose a body to describe the circle *ABC*; then at any point *A*, it is moving in the direction of the



tangent *AD*, and in this direction, by the first law of motion, it endeavours to proceed; also, since every point *D* in the tangent is without the circle, this tendency to move on in the direction of the tangent, is a tendency to recede from the center of motion; and the body will actually fly off, unless it is prevented by an adequate force.

The following experiment is given by Sir I. NEWTON to shew the effect of the centrifugal force, and to prove that it always accompanies an absolute circular motion.

Let a bucket, partly filled with water, be suspended

by a string, and turned round till the string is considerably twisted; then let the string be suffered to untwist itself, and thus communicate a circular motion to the vessel. At first the water remains at rest, and it's surface is smooth and undisturbed; but as it gradually acquires the motion of the bucket, the surface grows concave towards the center, and the water ascends up the sides, thus endeavouring to recede from the axis of motion; and this effect is observed gradually to increase with the absolute velocity of the water, till at length the water and the bucket are relatively at rest. When this is the case, let the bucket be suddenly stopped, and the absolute motion of the water will be gradually diminished by the friction of the vessel; the concavity of the surface is also diminished by degrees, and at length, when the water is again at rest, the surface becomes plane. Thus we find that the centrifugal force does not depend upon the relative, but upon the absolute motion, with which it always begins, increases, decreases, and disappears.

A single instance will be sufficient to shew the great utility of this conclusion in natural philosophy.

The diurnal rotation of the heavenly bodies may, as far as the appearance is concerned, be accounted for, either by supposing the heavens to revolve from east to west, and complete a revolution in twenty-four hours; or, the earth to revolve from west to east, in the same time: but the sensible diminution of gravity as we proceed towards the equator, and the oblate figure of the earth, which are the effects of a centrifugal force, prove that the appearance is to be ascribed to a real motion in the earth.

THE SECOND LAW OF MOTION.

(29.) *Motion, or change of motion, produced in a body, is proportional to the force impressed, and takes place in the direction in which the force acts.*

It has been seen in the preceding articles, that no motion or change of motion is ever produced in a body without *some* force impressed; we now assert that it cannot be produced without an *adequate* force; that when bodies act upon each other, the effects are not variable and accidental, but subject to general laws. Thus, whatever happens in one instance, will, under the same circumstances, happen again; and when any alteration takes place in the cause, there will be a corresponding and proportional alteration in the effect produced. Were not cause and effect thus connected with, and related to, each other, we could not pretend to lay down any general rules respecting the mutual actions of bodies; experiment could only furnish us with detached and isolated facts, wholly inapplicable on other occasions; and that harmony, which we cannot but observe and admire in the material world, would be lost.

In order to understand the meaning and extent of this law of motion, it will be convenient to distinguish it into two cases; and to point out such facts, under each head, as tend to establish it's truth.

1st. The same force, acting freely for a given time, will always produce the same effect, in the direction in which it acts.

Ex. 1. If a body, in one instance, fall perpendicularly through $16\frac{1}{12}$ feet in a second, and thus acquire a velocity which would carry it, uniformly, through $32\frac{1}{6}$ feet in that time, it will always, under the same circumstances, acquire the same velocity.

The effect produced is the same, whether the body begins to move from rest or not.

Ex. 2. If a body be projected perpendicularly downwards, the velocity of projection, measured in feet (Art. 11.), will, in one second, be increased by $32\frac{1}{6}$; and if it be projected perpendicularly upwards, it will, in one second, be diminished by that quantity.

Ex. 3. If a body be projected obliquely, gravity will still produce it's effect in a direction perpendicular to the horizon; and the body, which by it's inactivity would have moved uniformly forward in the line of it's first motion, will, at the end of one second, be found $16\frac{1}{12}$ feet below that line; having thus acquired a velocity of $32\frac{1}{6}$ feet per second, in the direction of gravity.

2d. If the force impressed be increased or diminished in any proportion, the motion communicated will be increased or diminished in the same proportion.

Ex. If a body descend along an inclined plane, the length of which is twice as great as it's height, the force which accelerates it's motion is half as great as the force of gravity; and, allowing for the effect of friction, and the resistance of the air, the velocity generated in any time is half as great as it would have been, had the body fallen, for the same time, by the whole force of gravity *.

(30.) In estimating the effect of any force, two circumstances are to be attended to; first, we must consider what force is actually impressed; for this alone can produce a change in the state of motion or quiescence of a body. Thus, the effect of a stream upon the floats of a water-wheel is not produced by the whole force of the stream, but by that part of it which arises from the excess of the velocity of the water above that of the wheel; and it is nothing, if they move with equal velocities. Secondly, we must

* The experiments which most satisfactorily prove the truth of this law of motion, are made with Mr. ATWOOD's machine, mentioned on a former occasion.

Let two weights, each of which is represented by $9m$, balance each other on this machine; and observe what velocity is generated in one second, when a weight $2m$ is added to either of them. Again, let the weights $8m$, $8m$, be sustained, as before, and add $4m$ to one of them, then the velocity generated in one second is twice as great as in the former instance. Since, therefore, the mass to be moved is the same in both cases, viz. $20m$ together with the inertia of the machine, it is manifest that when the moving force is doubled (Art. 23.), the momentum generated is also doubled, and, by altering the ratio of the weights, it may be shewn, in any other case, that the momentum communicated is proportional to the moving force impressed.

consider in what direction the force acts; and take that part of it, only, which lies in the direction in which we are estimating the effect. Thus, the force of the wind actually impressed upon the sails of a wind-mill, is not wholly employed in producing the circular motion; and therefore in calculating it's effect, in this respect, we must determine what part of the whole force acts in the direction of the motion.

In the following pages, we shall see a great variety of instances in which this method of estimating the effects of forces is applied; and the conclusions thus deduced, being found, without exception, to agree with experiment, we cannot but admit the truth of the principle.

(31.) Cor. Since the effect produced upon each other by two bodies, depends upon their relative velocity, it will always be the same whilst this remains unaltered, whatever be their absolute motions.

THE THIRD LAW OF MOTION.

(32.) *Action and reaction are equal, and in opposite directions.*

Matter not only perseveres in it's state of rest or uniform rectilinear motion, but also by it's inertia resists any change. Our experience with respect to this reaction, or opposition to the force impressed, is so constant and universal, that the very supposition of

it's non-existence appears to be absurd. For who can conceive a pressure without some support of that pressure? Who can suppose a weight to be raised without force or exertion? Thus far then we are assured by our senses, that whenever one body acts upon another, there is *some* reaction: The law farther asserts, that the reaction is *equal in quantity* to the action.

By *action*, we here understand moving force, which, according to the definition (Art. 22.), is measured by the momentum which is, or would be generated, in a given time; and to determine whether action and reaction, in this sense of the words, are equal or not, recourse must be had to experiment.

Take two similar and equal cylindrical pieces of wood, from one of which projects a small steel point; suspend them by equal strings, and let one of them descend through any arc and impinge upon the other at rest; then, by means of the steel point, the two bodies will move on together as one mass, and with a velocity equal to half the velocity of the impinging body. Thus the momentum, which is measured by the quantity of matter and velocity taken jointly, remains unaltered; or, as much momentum as is gained by the body struck, so much is taken from the momentum of the striking body, or communicated to it in the opposite direction.

If the striking body be loaded with lead, and thus made twice as heavy as the other, the common velocity after impact is found to be to the velocity of the impinging body :: 2 : 3; and because the joint mass after impact : quantity of matter in the striking body :: 3 : 2, the momentum after impact : momentum

before :: $3 \times 2 : 2 \times 3$, or in a ratio of equality, as in the former case.

In making experiments to establish this third law of motion, allowance must be made for the air's resistance; and care must be taken to obtain a proper measure of the velocity before and after impact. See Sir. I. NEWTON's Scholium to the Laws of Motion.

(33.) The third law of motion is not confined to cases of actual impact; the effects of pressures and attractions, in opposite directions, are also equal.

When two bodies sustain each other, the pressures in opposite directions must be equal, otherwise motion would ensue; and if motion be produced by the excess of pressure on one side, the case coincides with that of impact*.

When one body attracts another, it is itself also equally attracted, and as much momentum as is thus communicated to one body, will also be communicated to the other in the opposite direction.

A loadstone and a piece of iron, equal in weight, and floating upon similar and equal pieces of cork, approach each other with equal velocities, and therefore with equal momenta; and when they meet, or are kept asunder by any obstacle, they sustain each other by equal and opposite pressures.

(34.) COR. Since the action and reaction are equal

* The effects of pressure and impact are manifestly of the same kind, and produced in the same way; excess of pressure, on one side, produces momentum, and equal and opposite momenta support each other by opposite pressures.

Thus also pressures may be compared, either by comparing the weights which they sustain, or the momenta which they would generate under the same circumstances.

at every instant of time, the whole effect of the action in a finite time, however it may vary, is equal to the effect of the reaction; because the whole effects are made up of the effects produced in every instant.

SCHOLIUM.

(35.) These laws are the simplest principles to which motion can be reduced, and upon them the whole theory depends. They are not indeed self-evident, nor do they admit of accurate proof by experiment, on account of the great nicety required in adjusting the instruments, and making the experiments; and on account of the effects of friction, and the air's resistance, which cannot entirely be removed. They are however constantly, and invariably, suggested to our senses, and they agree with experiment as far as experiment can go; and the more accurately the experiments are made, and the greater care we take to remove all those impediments which tend to render the conclusions erroneous, the more nearly do the experiments coincide with these laws.

Their truth is also established upon a different ground; from these *general* principles innumerable *particular* conclusions have been deduced; sometimes the deductions are simple and immediate, sometimes they are made by tedious and intricate operations; yet they are all, without exception, consistent with each other and with experiment: it follows therefore that the principles, upon which the calculations are founded, are true *.

* Atwood on the Motions of Bodies, p. 359.

(36.) It will be necessary to remember, that the laws of motion relate, *immediately*, to the actions of particles of matter upon each other, or to those cases in which the whole mass may be conceived to be collected in a point; not to *all* the effects that may *eventually* be produced in the several particles of a system.

A body may have a rectilinear and rotatory motion given it at the same time, and it will retain both. The action also, or reaction, may be applied at a mechanical advantage or disadvantage, and thus they may produce, upon the whole, very different momenta: these effects depend upon principles which are not here considered, but which must be attended to in computing such effects.

SECTION III.

ON THE COMPOSITION AND RESOLUTION OF MOTION.

Prop. II.

(37.) *Two lines, which represent the momenta communicated to the same or equal bodies, will represent the spaces uniformly described by them in equal times; and conversely, the lines which represent the spaces uniformly described by them in equal times, will represent their momenta.*

The momenta of bodies may be represented by numbers, as was seen Art. 17; but in many cases it will be much more convenient to represent them by lines, because lines will express not only the quantities of the momenta, but also the directions in which they are communicated.

Any line drawn in the proper direction may be taken to represent one momentum; but to represent a second, a line, in the direction of the latter motion,

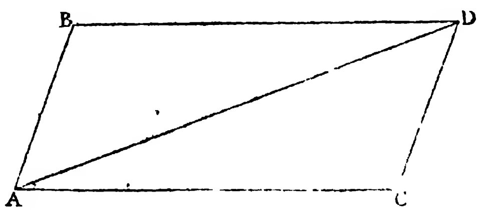
must be taken in the same proportion to the former line, that the second momentum has to the first.

Let two lines, thus taken, represent the momenta communicated to the same, or equal bodies; then since $M \propto V \times Q$ (Art. 17.), and Q is here given, $M \propto V$; therefore the lines, which represent the momenta, will also represent the velocities, or the spaces uniformly described in equal times. Again, if the lines represent the spaces uniformly described in equal times, they represent the velocities, and since Q is given, $V \propto QV \propto M$; therefore the lines represent the momenta.

Prop. III.

(38.) *Two uniform motions, which, when communicated separately to a body, would cause it to describe the adjacent sides of a parallelogram in a given time, will, when they are communicated at the same instant, cause it to describe the diagonal in that time; and the motion in the diagonal will be uniform.*

Let a motion be communicated to a body at A, which would cause it to move uniformly from A to B



in T'', and at the same instant, another motion which alone would cause it to move uniformly from A to C in T''; complete the parallelogram BC, and draw the

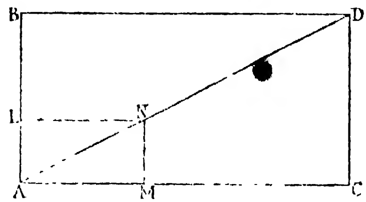
diagonal AD; then the body will arrive at the point D, in T'', having described AD with an uniform motion.

For the motion in the direction AC can neither accelerate nor retard the approach of the body to the line BD which is parallel to AC, (Art. 29. Ex. 3.); hence, the body will arrive at BD, in the same time that it would have done, had no motion been communicated to it in the direction AC, that is in T''. In the same manner, the motion in the direction AB can neither make the body approach to, nor recede from, CD; therefore, in consequence of the motion in the direction AC, it will arrive at CD in the same time that it would have done, had no motion been communicated in the direction AB, that is in T''. Hence it follows that, in consequence of the two motions, the body will be found both in BD and CD at the end of T'', and will therefore be found in D, the point of their intersection.

Also, since a body in motion continues to move uniformly forward in a right line, till it is acted upon by some external force (Art. 27.), the body A must have described the right line AD, with an uniform motion.

(39.) To illustrate this proposition, suppose a plane $ABDC$, as the deck of a ship, to be carried uniformly forward, and let the point A describe the line AC in T''; also, let a body move uniformly in this plane from A to B, in the same time. Complete the parallelogram BC, and draw the diagonal AD. Then at the end of T'' the body, by its own motion, will arrive at B; also by the motion of the plane, AB will be brought into the situation CD, and the point B

will coincide with D; therefore the body will upon the whole, at the end of T'', be found in D. In any other time t'', let the point A be carried from A to



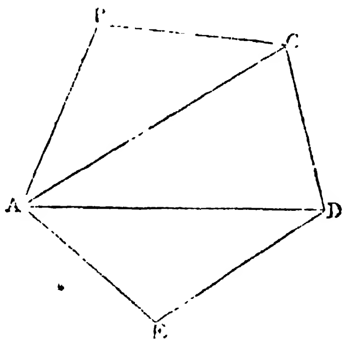
M by the motion of the plane, and the body from A to L by it's own motion; complete the parallelogram $ALNM$, and join AN; then, as in the preceding case, the body will, at the end of t'', be found in N; and since the motions in the directions AC, AB are uniform, $AC : AM :: T : t :: AB : AL$ (Art. 13.); that is, the sides of the parallelograms, about the common angle LAM, are proportional, and consequently the parallelograms are about the same diagonal AD (Euc. 26. vi.); therefore the body at the end of any time t'' will be found in the diagonal AD. It will also move uniformly in the diagonal; for, from the similar triangles AMN, ACD, we have $AD : AN :: AC : AM :: T : t$, or the spaces described are proportional to the times. (See Art. 10.)

(40.) Cor. 1. The reasoning in the last article is applicable to the motion of a point.

(41.) Cor. 2. If two sides of a triangle, AB, BD, taken in order, represent the spaces over which two uniform motions would, separately, carry a body in a given time; when these motions are communicated at the same instant to the body at A, it will describe the third side AD, uniformly, in that time.

For, if the parallelogram *BC* be completed, the same motion, which would carry a body uniformly from *B* to *D*, would, if communicated at *A*, carry it in the same manner from *A* to *C*; and in consequence of this motion, and of the motion in the direction *AB*, the body would uniformly describe the diagonal *AD*, which is the third side of the triangle *ABD*.

(42.) Cor. 3. In the same manner, if the lines *AB*, *BC*, *CD*, *DE*, taken in order, represent the spaces over which any uniform motions would, separately,



carry a body, in a given time, these motions, when communicated at the same instant, will cause the body to describe the line *AE* which completes the figure, in that time; and the motion in this line will be uniform.

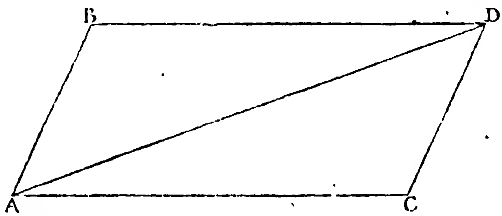
(43.) Cor. 4. If *AD* represent the uniform velocity of a body, and any parallelogram *ABDC* (Art. 38.), be described about it, the velocity *AD* may be supposed to arise from the two uniform velocities *AB*, *AC*, or *AB*, *BD*; and if one of them, *AB*, be by any means taken away, the velocity remaining will be represented by *AC* or *BD*. (See Art. 11.)

(44.) Def. A force is said to be *equivalent* to any number of forces, when it will, *singly*, produce the same effect that the others produce *jointly*, in any given time.

Prop. IV.

(45.) *If the adjacent sides of a parallelogram represent the quantities and directions of two forces, acting at the same time upon a body, the diagonal will represent one equivalent to them both.*

Let AB, AC represent two forces acting upon a body at A, then they represent the momenta communicated to it in those directions (Art. 22.), and conse-



quently the spaces which it would uniformly describe in equal times (Art. 37). Complete the parallelogram CB, and draw the diagonal AD; then, by the last proposition, AD is the space uniformly described in the same time, when the two motions are communicated to the body at the same instant; and since AB, AC, and AD, represent the spaces uniformly described by the same body, in equal times, they represent the momenta, and therefore the forces acting in those directions; that is, the forces AB, AC*, acting at the same time, produce a force which is represented, in quantity and direction, by AD.

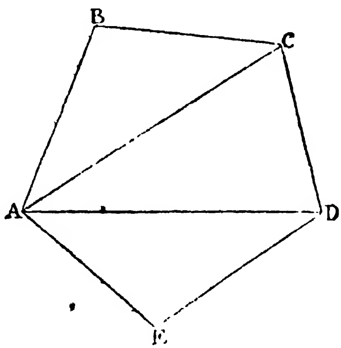
* In this, and many other cases, where forces are represented by lines, the lines are used, for the sake of conciseness, to express the forces which they represent.

DEF. The force represented by *AD* is said to be *compounded* of the two, *AB*, *AC*.

(46.) COR. 1. If two sides of a triangle, taken in order, represent the quantities and directions of two forces, the third side will represent a force equivalent to them both.

For a force represented by *BD*, acting at *A*, will produce the same effect that the force *AC*, which is equal to it and in the same direction, will produce; and *AB*, *AC*, are equivalent to *AD*; therefore *AB*, *BD* are also equivalent to *AD*.

(47.) COR. 2. If any lines *AB*, *BC*, *CD*, *DE*,

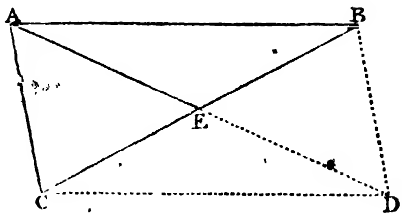


taken in order, represent the quantities and directions of forces communicated at the same time to a body at *A*, the line *AE*, which completes the figure, will represent a force equivalent to them all.

For the two *AB*, *BC* are equivalent to *AC*; also, *AC*, *CD*, that is, *AB*, *BC*, *CD*, are equivalent to *AD*; in the same manner *AD*, *DE*, that is, *AB*, *BC*, *CD*, and *DE*, are equivalent to *AE*.

(48.) COR. 3. Let *AB* and *AC* represent the quantities and directions of two forces, join *BC* and

draw AE bisecting it in E, then will $2AE$ represent

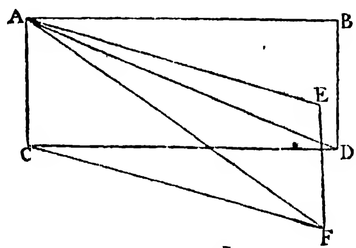


a force equivalent to them both.

For, if the parallelogram be completed, since the diagonals bisect each other, AD, which represents a force equivalent to AB and AC, is equal to $2AE$.

(49.) Cor. 4. If the angle at which two given forces act be diminished, the compound force is increased.

Let AB, AC be the two given forces; complete



the parallelogram $ABDC$ and draw the diagonal AD, this represents the compound force. In the same manner, if AE be taken equal to AB, and AE, AC, represent the two forces, then AF the diagonal of the parallelogram $AEFC$, represents the compound force; and since the angle BAC is greater than the angle EAC, ACD which is the supplement of the former, is less than ACF the supplement of the latter; also, $CF = AE = AB = CD$; therefore in the two triangles ACD,

ACF, the sides *AC*, *CD* are respectively equal to *AC*, *CF*, and the $\angle ACD$ is less than the $\angle ACF$; consequently *AD* is less than *AF* (Euc. 24. i).

(50.) Cor. 5. Two given forces produce the greatest effect when they act in the same direction, and the least when they act in opposite directions; for, in the former case, the diagonal *AF* becomes equal to the sum of the sides *AC*, *CF*; and in the latter, to their difference.

(51.) Cor. 6. Two forces cannot keep a body at rest, unless they are equal and in opposite directions.

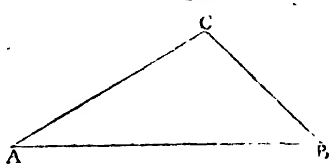
For this is the only case in which the diagonal, representing the compound force, vanishes.

(52.) Cor. 7. In the composition of forces, force is lost; for the forces represented by the two sides *AB*, *BD* (Art. 45.), by composition produce the force represented by *AD*; and the two sides *AB*, *BD*, of a triangle, are greater than the third side *AD*.

Prop. V.

(53.) *If a body, at rest, be acted upon at the same time by three forces which are represented in quantity and direction by the three sides of a triangle, taken in order, it will remain at rest.*

Let *AB*, *BC*, and *CA*, represent the quantities and



directions of three forces acting at the same time upon a body at *A*; then since *AB* and *BC* are equivalent to *AC* (Art. 46.); *AB*, *BC* and *CA* are equivalent to

AC and *CA*; but *AC* and *CA*, which are equal and in opposite directions, keep the body at rest; therefore *AB*, *BC*, and *CA*, will also keep the body at rest.

Prop. VI.

(54.) *If a body be kept at rest by three forces, and two of them be represented in quantity and direction by two sides* AB, BC*, *of a triangle, the third side, taken in order, will represent the quantity and direction of the other force.*

Since *AB*, *BC* represent the quantities and directions of two of the forces, and *AB*, *BC* are equivalent to *AC*, the third force must be sustained by *AC*; therefore *CA* must represent the quantity and direction of the third force (Art. 51).

(55.) Cor. If three forces keep a body at rest, they act in the same plane; because the three sides of a triangle are in the same plane (Euc. 2. xi).

Prop. VII.

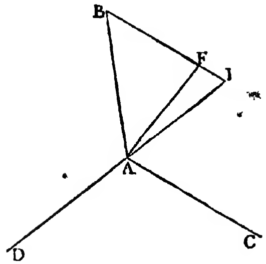
(56.) *If a body be kept at rest by three forces, acting upon it at the same time, any three lines, which are in the directions of these forces, and form a triangle, will represent them.*

Let three forces, acting in the directions *AB*, *AC*, *AD*, keep the body *A* at rest; then *AB*, *AC*, *AD* are in the same plane (Art. 55). In *AB* take any point, *B*, and through *B* draw *BI* parallel to *AC*, meeting *DA* produced in *I*; then will *AB*, *BI*, and *IA* represent the three forces.

For *AB* being taken to represent the force in that direction, if *BI* do not represent the force in the direc-

* Fig. Art. 53.

tion *AC* or *BI*, let *BF* be taken to represent it; join



AF; then since three forces keep the body at rest, and *AB*, *BF* represent the quantities and directions of two of them, *FA* will represent the third (Art. 54.), that is, *FA* is in the direction *AD*, which is impossible (Euc. 11. i. Cor.); therefore *BI* represents the force in the direction *AC*; and consequently *IA* represents the third force (Art. 54).

Any three lines, respectively parallel to *AB*, *BI*, *IA*, and forming a triangle, will be proportional to the sides of the triangle *ABI*, and therefore proportional to the three forces.

(57.) Cor. 1. If a body be kept at rest by three forces, any two of them are to each other inversely as the sines of the angles which the lines of their directions make with the direction of the third force.

Let *ABI* be a triangle whose sides are in the directions of the forces; then these sides represent the forces; and *AB* : *BI* :: sin. *BIA* : sin. *BAI* :: sin. *IAC* : sin. *BAI* :: sin. *CAD* : sin. *BAD*.

(58.) Cor. 2. If a body, at rest, be acted upon at the same time by three forces, in the directions of the sides of a triangle taken in order, and any two of them

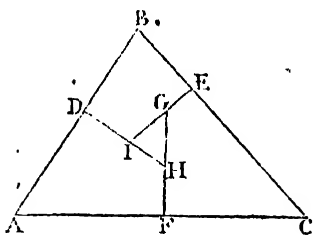
be to each other inversely as the sines of the angles which their directions make with the direction of the third, the body will remain at rest.

For, in this case, the forces will be proportional to the three sides of the triangle, and consequently they will sustain each other (Art. 53).

Prop. VIII.

(59.) *If a body be kept at rest by three forces, and lines be drawn at right angles to the directions in which they act, forming a triangle, the sides of this triangle will represent the quantities of the forces.*

Let *AB, BC, CA* be the directions in which the forces act; and let them form the triangle *ABC*; then the lines *AB, BC, CA*, will represent the forces



(Art. 56). Draw the perpendiculars *DH, EI, FG*, forming a triangle *GHI*; then since the four angles of the quadrilateral figure *ADHF* are equal to four right angles, and the angles at *D* and *F* are right angles, the remaining angles *DHF, DAF* are equal to two right angles, or to the two angles *DHF, DHG*; consequently, the angle *DAF* is equal to the angle *IHG*. In the same manner, it may be shewn, that the angles *ABC, BCA* are respectively equal to

GIH, HGI; therefore the triangles *ABC* and *GHI* are equiangular; hence, the sides about their equal angles being proportional, the forces, which are proportional to the lines *AB, BC, CA*, are proportional to the lines *HI, IG, GH*.

COR. If the lines *DH, EI, FG* be equally inclined to the lines *DB, EC, FA*, and form a triangle *GHI*, the sides of this triangle will represent the quantities of the forces.

PROP. IX.

(60.) *If any number of forces, represented in quantity and direction by the sides of a polygon, taken in order, act at the same time upon a body at rest, they will keep it at rest.*

Let *AB, BC, CD, DE*, and *EA* (Fig. Art. 47.), represent the forces; then since *AB, BC, CD* and *DE* are equivalent to *AE* (Art. 47.); *AB, BC, CD, DE*, and *EA*, are equivalent to *AE* and *EA*; that is, they will keep the body at rest.

PROP. X.

(61.) *If any number of lines, taken in order, represent the quantities and directions of forces which keep a body at rest, these lines will form a polygon.*

Let *AB, BC, CD* and *DE* represent forces which keep a body at rest (Fig. Art. 47.); then the point *E* coincides with *A*. If not, join *AE*, then *AB, BC, CD*, and *DE*, are equivalent to *AE*; and the body will be put in motion by a single force *AE*,

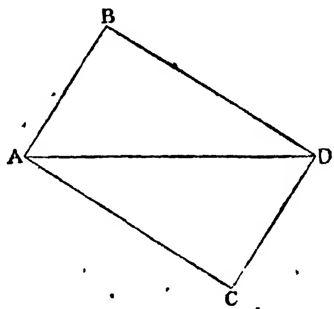
which is contrary to the supposition; therefore the point E coincides with A, and the lines form a polygon.

This and the last proposition are true when the forces act in different planes.

Prop. XI.

(62.) *A single force may be resolved into any number of forces.*

Since the single force AD is equivalent to the two, AB, BD, it may be conceived to be made up of, or resolved into, the two, AB, BD. The force AD may



therefore be resolved into as many pairs of forces as there can be triangles described upon AD, or parallelograms about it. Also AB, or BD, may be resolved into two; and, by proceeding in the same manner, the original force may be resolved into any number of others.

(63.) Cor. 1. If two forces are together equivalent to AD, and AB be one of them, BD is the other.

(64.) Cor. 2. If the force AD be resolved into the two, AB, BD, and AB be wholly lost, or destroyed,

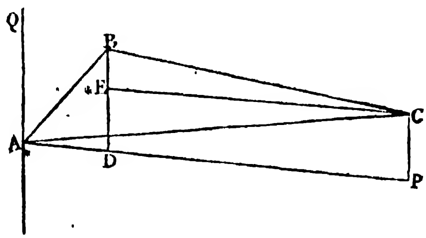
the effective part of AD is represented in quantity and direction by BD.

(65.) Cor. 3. In the resolution of forces, the whole quantity of force is increased. For the force represented by AD is resolved into the two AB, BD which are together greater than AD (Euc. 20. i).

Prop. XII.

(66.) *The effects of forces, when estimated in given directions, are not altered by composition or resolution.*

Let two forces AB, BC, and the force AC which is equivalent to them both, be estimated in the directions



AP, AQ. Draw BD, CP parallel to AQ; and CE parallel to AP. Then the force AB is equivalent to the two AD, DB; of which AD is in the direction AP, and DB in the direction AQ; in the same manner, BC is equivalent to the two BE, EC; the former of which is in the direction BD or QA, and the latter in the direction EC or AP; therefore the forces AB, BC, when estimated in the directions AP, AQ, are equivalent to AD, EC, DB, and BE; or, AD, DP,

DB and *BE*, because *EC* is equal to *DP*; and since *DB* and *BE* are in opposite directions, the part *EB* of the force *DB* is destroyed by *BE*; consequently, the forces are equivalent to *AP*, *DE*, or *AP*, *PC*. Also *AC*, when estimated in the proposed directions, is equivalent to *AP*, *PC*; therefore the effective forces in the directions *AP*, *AQ* are the same, whether we estimate *AB* and *BC*, in those directions, or *AC*, which is equivalent to them.

(67.) Cor. When *AP* coincides with *AC*, *EC* also coincides with it, and *D* coincides with *E*. In this case the forces *DB*, *BE* wholly destroy each other; and thus, in the composition of forces, force is lost.

SECTION IV.

ON THE MECHANICAL POWERS.

(68.) The mechanical powers are the most simple instruments used for the purpose of supporting weights, or communicating motion to bodies, and by the combination of which, all machines, however complicated, are constructed.

These powers are six in number, viz. the *lever*; the *wheel and axle*; the *pulley*; the *inclined* plane; the *wedge*; and the *screw*.

Before we enter upon a particular description of these instruments, and the calculation of their effects, it is necessary to premise, that when any forces are applied to them, they are themselves supposed to be at rest; and consequently, that they are either without weight, or that the parts are so adjusted as to sustain each other. They are also supposed to be perfectly smooth; no allowance being made for the effects of adhesion.

When two forces act upon each other by means of any machine, one of them is, for the sake of distinction, called the *power*, and the other the *weight*.

ON THE LEVER.

(69.) Def. The *Lever* is an inflexible rod, moveable, in one plane, upon a point which is called the *fulcrum*, or *center of motion*.

The power and weight are supposed to act in the plane in which the lever is moveable round the fulcrum, and tend to turn it in opposite directions.

(70.) The properties of the lever cannot be deduced immediately from the propositions laid down in the last section, because the forces acting upon the lever are not applied at a point, which is always supposed to be the case in the composition and resolution of forces; they may however be derived from the following principles, the truth of which will readily be admitted.

Ax. 1. *If two weights balance each other upon a straight lever, the pressure upon the fulcrum is equal to the sum of the weights, whatever be the length of the lever* *.

Ax. 2. *If a weight be supported upon a lever which rests on two fulcrums, the pressure upon the fulcrums is equal to the whole weight*:

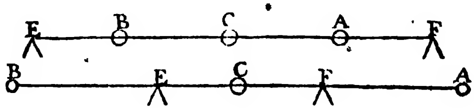
Ax. 3. *Equal forces, acting perpendicularly at the extremities of equal arms of a lever, exert the same effort to turn the lever round.*

* The effect produced by the gravity of the lever is not taken into consideration, unless it be expressly mentioned.

Prop. XIII.

(71.) *If two equal weights act perpendicularly upon a straight lever, the effort to put it in motion, round any fulcrum, will be the same as if they acted together at the middle point between them.*

Let A and B be two equal weights, acting perpendicularly upon the lever FB, whose fulcrum is F.



Bisect AB in C; make $CE = CF$; and at E suppose another fulcrum to be placed.

Then since the two weights A and B are supported by E and F, and these fulcrums are similarly situated with respect to the weights, each sustains an equal pressure; and therefore the weight sustained by E is equal to half the sum of the weights. Now let the weights A and B be placed at C, the middle point between A and B, and consequently the middle point between E and F; then since E and F support the whole weight C, and are similarly situated with respect to it, the fulcrum E supports half the weight; that is, the pressure upon E is the same, whether the weights are placed at A and B, or collected in C, the middle point between them; and therefore, the effort to put the lever in motion round F, is the same on either supposition.

(72.) Cor. If a weight be formed into a cylinder *AB* (Fig. Art. 73.) which is every where of the same density, and placed parallel to the horizon, the effort of any part *AD*, to put the whole in motion round *C*, is the same as if this part were collected at *E*, the middle point of *AD*.

For the weight *AD* may be supposed to consist of pairs of equal weights, equally distant from the middle point.

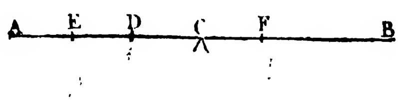
What is here affirmed of weights, is true of any forces which are proportional to the weights, and act in the same directions.

Prop. XIV.

(73.) *Two weights, or two forces, acting perpendicularly upon a straight lever, will balance each other, when they are reciprocally proportional to their distances from the fulcrum.*

Case 1. When the weights act on *contrary* sides of the fulcrum.

Let x and y be the two weights, and let them be formed into the cylinder *AB*, which is every where of the same density. Bisect *AB* in *C*; then this cylinder



will balance itself upon the fulcrum *C* (Art. 72). Divide *AB* into two parts in *D*, so that $AD : DB :: x : y$, and the weights of *AD* and *DB* will be respectively x and y; bisect *AD* in *E* and *DB* in *F*; then since *AD* and *DB* keep the lever at rest, they will keep it at rest

when they are collected at E and F (Art. 72.); that is, x, when placed at E, will balance y, when placed at F; and $x : y :: AD : BD :: AB - BD : AB - AD :: 2CB - 2BF : 2AC - 2AE :: 2CF : 2CE :: CF : CE$.

CASE 2. When the two forces act on the *same* side of the center of motion.

Let AB be a lever whose fulcrum is C; A and B two weights acting perpendicularly upon it; and let $A : B :: BC : AC$; then these weights will balance each other, as appears by the former Case. Now suppose a power sufficient to sustain a weight equal to the sum of the weights A and B, to be applied at C, in a direc-

tion opposite to that in which the weights act; then will this power supply the place of the fulcrum (Art. 70. Ax. 1.); also, a fulcrum placed at A, or B, and sustaining a weight A, or B, will supply the place of the body there, and the equilibrium will remain. Let B be the center of motion; then we have a straight lever whose center of motion is B, and the two forces A and $A+B$, acting perpendicularly upon it at the points A and C, sustain each other; also, $A : B :: BC : AC$; therefore $A : A+B :: BC : BA$.

(74.) COR. 1. If two weights, or two forces, acting perpendicularly on the arms of a straight lever, keep each other in equilibrio, they are inversely as their distances from the center of motion.

For the weights will balance when they are in that proportion, and if the proportion be altered by increasing or diminishing one of the weights, it's effort

to turn the lever round will be altered, or the equilibrium will be destroyed.

(75.) Cor. 2. Since $A : B :: BC : AC$ when there is an equilibrium upon the lever AB, whose fulcrum is C, by multiplying extremes and means, $A \times AC = B \times BC$.

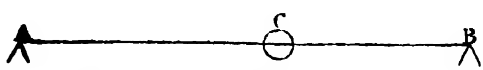
(76.) Cor. 3. When the power and weight act on the same side of the fulcrum, and keep each other in equilibrio, the weight sustained by the fulcrum is equal to the difference between the power and the weight.

(77.) Cor. 4. In the common balance, the arms of the lever are equal; consequently, the power and weight, or two weights, which sustain each other, are equal. In the false balance, one arm is longer than the other; therefore the weight, which is suspended at this arm, is proportionally less than the weight which it sustains at the other.

(78.) Cor. 5. If the same body be weighed at the two ends of a false balance, it's true weight is a mean proportional between the apparent weights.

Call the true weight x, and the apparent weights, when it is suspended at A and B, a and b respectively; then $a : x :: AC : BC$, and $x : b :: AC : BC$; therefore $a : x :: x : b$.

(79.) Cor. 6. If a weight C be placed upon a lever which is supported upon two props A and B in an

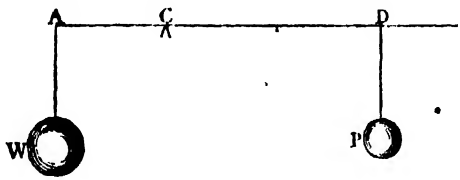


horizontal position, the pressure upon A : the pressure upon $B :: BC : AC$.

For if B be conceived to be the fulcrum, we have this proportion, the weight sustained by A : the weight

C :: BC : AB; in the same manner, if A be considered as the fulcrum, then the weight C : the weight sustained by B :: AB : CA; therefore, *ex æquo*, the weight sustained by A : the weight sustained by B :: BC : AC.

(80.) Cor. 7. If a given weight P be moved along the graduated arm of a straight lever, the weight W,



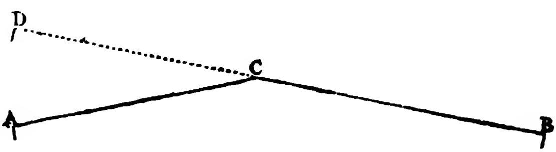
which it will balance at A, is proportional to CD, the distance at which the given weight acts.

When there is an equilibrium, $W \times AC = P \times DC$ (Art. 75.); and AC and P are invariable; therefore $W \propto DC$ (*Alg.* Art. 199.)

Prop. XV.

(81.) *If two forces, acting upon the arms of any lever, keep it at rest, they are to each other inversely as the perpendiculars drawn from the center of motion to the directions in which the forces act.*

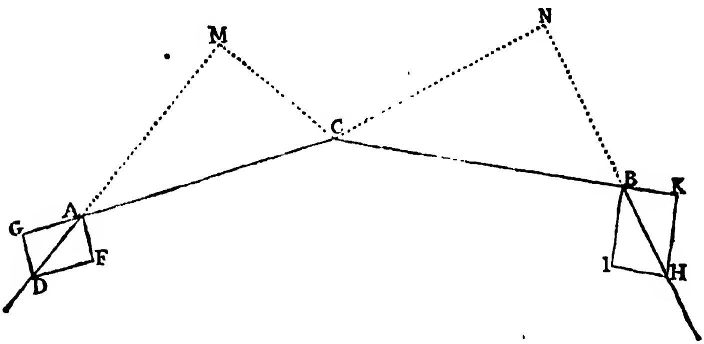
Case 1. Let two forces, A and B, act *perpendicularly* upon the arms CA, CB, of the lever ACB whose fulcrum is C, and keep each other at rest. Produce



BC to D, and make $CD = CA$; then the effort of A

to move the lever round C, will be the same, whether it be supposed to act perpendicularly at the extremity of the arm CA, or CD (Art. 70. Ax. 3.); and on the latter supposition, since there is an equilibrium, $A : B :: CB : CD$ (Art. 74.); therefore $A : B :: CB : CA$.

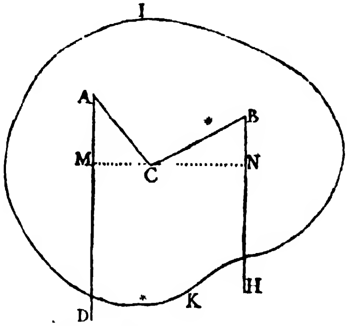
Case 2. When the directions AD, BH, in which the forces act, are not perpendicular to the arms. Take



AD and BH, to represent the forces; draw CM and CN at right angles to those directions; also draw AF perpendicular, and DF parallel to AC, and complete the parallelogram GF; then the force AD is equivalent to the two AF, AG, of which, AG acts in the direction of the arm, and therefore can have no effect in causing, or preventing any angular motion in the lever about C. Let BH be resolved, in the same manner, into the two BI, BK, of which BI is perpendicular to, and BK in the direction of the arm CB; then BK will have no effect in causing, or preventing any angular motion in the lever about C; and since the lever is kept at rest, AF and BI, which produce this effect, and act perpendicularly upon the arms, are to each other, by the 1st case, inversely as the arms; that is, $AF : BI :: CB : CA$, or $AF \times CA = BI \times CB$. Also in the similar

triangles ADF, ACM, $AF : AD :: CM : CA$, and $AF \times CA = AD \times CM$; in the same manner, $BI \times CB = BH \times CN$; therefore $AD \times CM = BH \times CN$, and $AD : BH :: CN : CM$.

(82.) COR. 1. Let a body IK be moveable about the center C, and two forces act upon it at A and B,



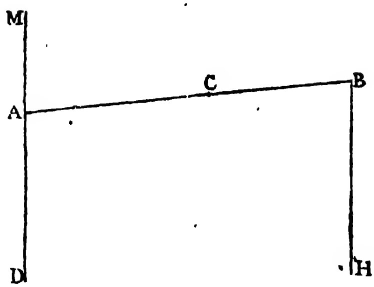
in the directions AD, BH, which coincide with the plane ACB; join AC, CB; then this body may be considered as a lever ACB, and drawing the perpendiculars CM, CN, there will be an equilibrium, when the force acting at A : the force acting at B :: $CN : CM$*.

(83.) COR. 2. The effort of the force A, to turn the lever round, is the same, at whatever point in the direction MD it is applied; because the perpendicular CM remains the same.

(84.) COR. 3. Since $CA : CM ::$ rad. : sin. CAM, $CM = \frac{CA \times \text{sin. } CAM}{\text{rad.}}$; and in the same manner, $CN = \frac{CB \times \text{sin. } CBN}{\text{rad.}}$; therefore, when there is an equilibrium, the power at A : the weight at B $\frac{CB \times \text{sin. } CBN}{\text{rad.}} : \frac{CA \times \text{sin. } CAM}{\text{rad.}} :: CB \times \text{sin. } CBN : CA \times \text{sin. } CAM$.

* See Art. 74.

(85.) Cor. 4. If the lever *ACB* be straight, and



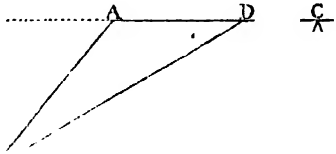
the directions *AD*, *BH*, parallel, $A : B :: BC : AC$; because, in this case, sin. $CAM =$ sin. CBH.

Hence also, $A \times AC = B \times BC$.

(86.) Cor. 5. If two weights balance each other upon a straight lever in any one position, they will balance each other in any other position of the lever; for the weights act in parallel directions, and the arms of the lever are invariable.

(87.) Cor. 6. If a man, balanced in a common pair of scales, press upwards by means of a rod, against any point in the beam, except that from which the scale is suspended, he will preponderate.

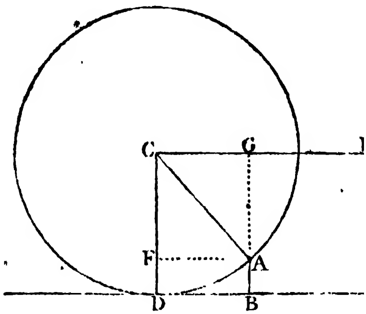
Let the action upwards take place at *D*, then the scale, by the reaction downwards, will be brought



into the situation *E*; and the effect will be the same as if *DA*, *AE*, *DE*, constituted one mass; that is,

drawing EF perpendicular to CA produced, as if the scale were applied at F (Art. 83.); consequently the weight, necessary to maintain the equilibrium, is greater than if the scale were suffered to hang freely from A, in the proportion of $CF : CA$ (Art. 80.).

(88.) Cor. 7. Let AD represent a wheel, bearing a weight at it's center C; AB an obstacle over which

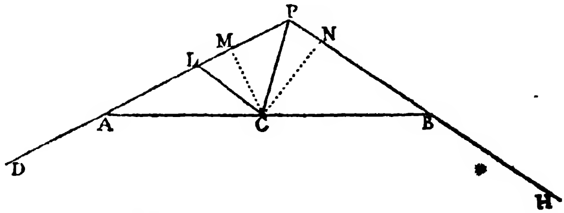


it is to be moved by a force acting in the direction CE; join CA, draw CD perpendicular to the horizon, and from A draw AG, AF, at right angles to CE, CD. Then CA may be considered as a lever whose center of motion is A, CD the direction in which the weight acts, and CE the direction in which the power is applied; and there is an equilibrium on this lever when the power : the weight :: $AF : AG$.

Supposing the wheel, the weight, and the obstacle given, the power is the least when AG is the greatest; that is, when CE is perpendicular to CA, or parallel to the tangent at A.

(89.) Cor. 8. Let two forces acting in the directions AD, BH, upon the arms of the lever ACB, keep each other in equilibrio; produce DA and HB till they meet in P; join CP, and draw CL parallel

to PB; then will PL, LC represent the two forces,

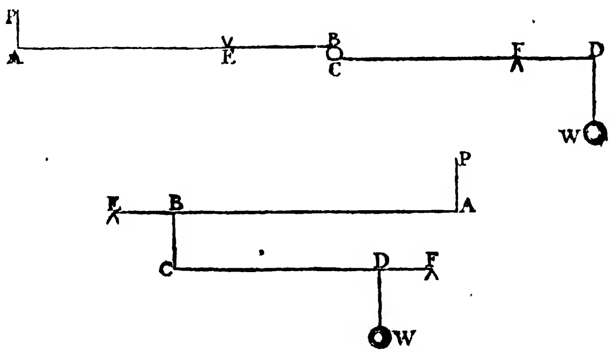


and PC the pressure upon the fulcrum.

For, if PC be made the radius, CM and CN are the sines of the angles CPM, CPN, or CPL, PCL; and $PL : LC :: \text{sin.}\ PCL : \text{sin.}\ LPC : CN : CM$; therefore PL, LC, represent the quantities and directions of the two forces, which may be supposed to be applied at P (Art. 83.), and which are sustained by the reaction of the fulcrum; consequently, CP represents the quantity and direction of that reaction (Art. 54.), or PC represents the pressure upon the fulcrum.

Prop. XVI.

(90.) *In a combination of straight levers,* AB, CD, *whose centers of motion are* E *and* F, *if they act perpendicularly upon each other, and the directions in which the power and weight are applied be also perpendicular to the arms, there is an equilibrium when* P : W :: EB × FD : EA × FC.



For, the power at A : the weight at B, or C :: EB : EA; and the weight at C : the weight at D :: FD : FC; therefore, $P : W :: EB \times FD : EA \times FC$.

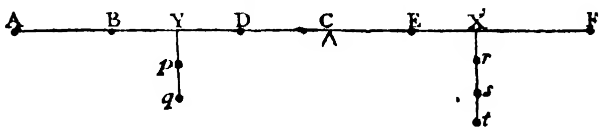
By the same method we may find the proportion between the power and the weight, when there is an equilibrium, in any other combination of levers.

(91.) Cor. If E and F be considered as the power and weight, A and D the centers of motion, we have, as before, $E : F :: FD \times BA : AE \times CD$. Hence the pressure upon E : the pressure upon F :: $FD \times BA$: $AE \times CD$.

Prop. XVII.

(92.) *Any weights will keep each other in equilibrio on the arms of a straight lever, when the products, which arise from multiplying each weight by it's distance from the fulcrum, are equal, on each side of the fulcrum.*

The weights A, B, D, and E, F, will balance each other upon the lever AF whose fulcrum is C, if $A \times AC + B \times BC + D \times DC = E \times EC + F \times FC$.

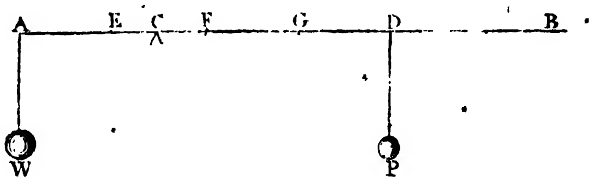


In CF take any point X, and let the weights r, s, t, placed at X, balance respectively, A, B, D; then $A \times AC = r \times XC$; $B \times BC = s \times XC$; $D \times DC = t \times XC$, (Art. 85.); or, $A \times AC + B \times BC + D \times DC = \overline{r+s+t} \times XC$. In the same manner, let p and q, placed at Y,

balance respectively, E and F; then $\overline{p+q} \times YC = E \times EC + F \times FC$; but by the supposition $A \times AC + B \times BC + D \times DC = E \times EC + F \times FC$; therefore $\overline{r+s+t} \times XC = \overline{p+q} \times YC$, and the weights r, s, t, placed at X, balance the weights p, q, placed at Y; also A, B, D, balance the former weights, and E, F, the latter; consequently A, B, D, will balance E and F.

(93.) Cor. 1. If the weights do not act in parallel directions, instead of the distances we must substitute the perpendiculars, drawn from the center of motion, upon the directions. (See Art. 81.)

(94.) Cor. 2. In Art. 80. the lever is supposed to be without weight, or the arms AC, CD to balance each other: In the formation of the common *steel-yard* the longer arm CB is heavier than CA, and allowance must be made for this excess. Let the

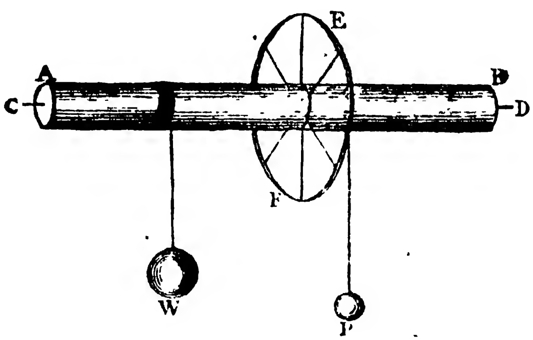


moveable weight P, when placed at E, keep the lever at rest; then when W and P are suspended upon the lever, and the whole remains at rest, W sustains P, and also a weight which would support P when placed at E; therefore $W \times AC = P \times DC + P \times EC = P \times DE$; and since AC and P are invariable, $W \propto ED$; the graduation must therefore begin from E; and if P,

when placed at *F*, support a weight of one pound at *A*, take *FG*, *GD*, &c. equal to each other, and to *EF*, and when *P* is placed at *G* it will support two pounds; when at *D* it will support three pounds, &c.

ON THE WHEEL AND AXLE.

(95.) The *wheel and axle* consists of two parts, a cylinder *AB* moveable about it's axis *CD*, and a circle



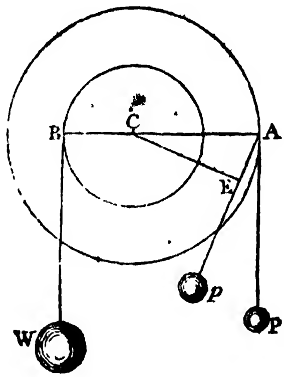
EF so attached to the cylinder that the axis *CD* passes through it's center, and is perpendicular to it's plane.

The power is applied at the circumference of the wheel, usually in the direction of a tangent to it, and the weight is raised by a rope which winds round the axle in a plane at right angles to the axis.

PROP. XVIII.

(96.) *There is an equilibrium upon the wheel and axle, when the power is to the weight, as the radius of the axle to the radius of the wheel.*

The effort of the power to turn the machine round the axis, must be the same at whatever point in the axle the wheel is fixed; suppose it to be removed, and placed in such a situation that the power and weight may act



in the same plane, and let CA, CB, be the radii of the wheel and axle, at the extremities of which the power and weight act; then the machine becomes a lever ACB, whose center of motion is C; and since the radii CA, CB, are at right angles to AP and BW, we have $P : W :: CB : CA$ (Art. 82).

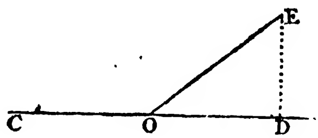
(97.) COR. 1. If the power act in the direction Ap, draw CE perpendicular to Ap, and there will be an equilibrium when $P : W :: CB : CE$ (Art. 82).

The same conclusion may also be obtained by resolving the power into two, one perpendicular to AC, and the other parallel to it.

(98.) Cor. 2. If $2R$ be the thickness of the ropes by which the power and weight act, there will be an equilibrium when $P : W :: CB+R : CA+R$, since the power and weight must be supposed to be applied in the axes of the ropes.

The ratio of the power to the weight is greater in this case than the former; for if any quantity be added to the terms of a ratio of less inequality, that ratio is increased (*Alg.* Art. 162.).

(99.) Cor. 3. If the plane of the wheel be inclined to the axle at the angle EOD, draw ED perpendicular



to CD; and considering the wheel and axle as one mass, there is an equilibrium when $P : W ::$ the radius of the axle : ED.

(100.) Cor. 4. In a combination of wheels and axles, where the circumference of the first axle is applied to the circumference of the second wheel, by means of a string, or by tooth and pinion, and the second axle to the third wheel, &c. there is an equilibrium when $P : W ::$ the product of the radii of all the axles : the product of the radii of all the wheels. (See Art. 90.)

(101.) Cor. 5. When the power and weight act in parallel directions, and on *opposite* sides of the axis, the pressure upon the axis is equal to their *sum*; and when they act on the *same* side, to their *difference*. In other cases the pressure may be estimated by Art. 89.

ON THE PULLEY.

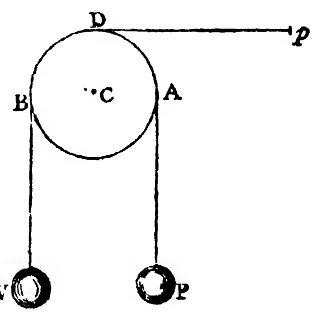
(102.) Def. A *Pulley* is a small wheel moveable about it's center, in the circumference of which a groove is formed to admit a rope or flexible chain.

The pulley is said to be *fixed*, or *moveable*, according as the center of motion is fixed or moveable.

Prop. XIX.

(103.) *In the single fixed pulley, there is an equilibrium, when the power and weight are equal.*

Let a power and weight *P*, *W*, equal to each other, act by means of a perfectly flexible rope *PDW*, which passes over the fixed pulley *ADB*; then, whatever



force is exerted at *D* in the direction *DAP*, by the power, an equal force is exerted by the weight in the direction *DBW*; these forces will therefore keep each other in equilibrio.

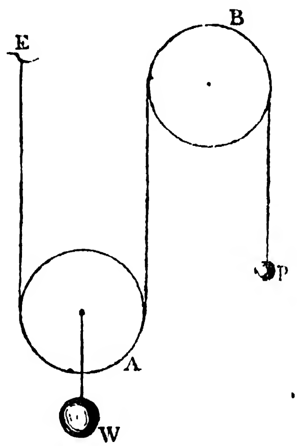
COR. 1. Conversely, when there is an equilibrium, the power and weight are equal*.

COR. 2. The proposition is true in whatever direction the power is applied; the only alteration made, by changing it's direction, is in the pressure upon the center of motion. (See.Art. 106.)

PROP. XX.

(104.) *In the single moveable pulley, whose strings are parallel, the power is to the weight as* 1 *to* 2†.

A string fixed at *E*, passes under the moveable pulley *A*, and over the fixed pulley *B*; the weight is



annexed to the center of the pulley *A*, and the power is applied at *P*. Then since the strings *EA*, *BA* are

* See Art. 74.

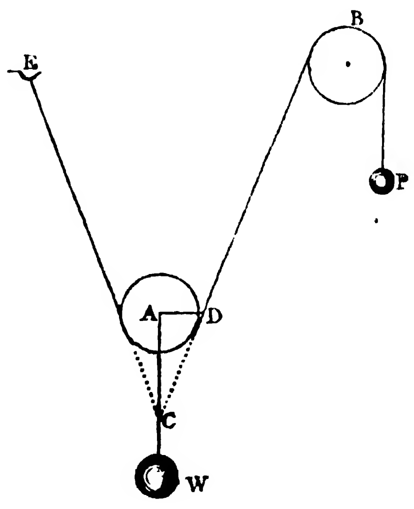
† In this and the following propositions, the power and weight are supposed to be in equilibrio.

in the direction in which the weight acts, they exactly sustain it; and they are equally stretched in every point, therefore they sustain it equally between them; or each sustains half the weight. Also, whatever weight *AB* sustains, *P* sustains (Prop. xix. Cor. 1.), therefore $P : W :: 1 : 2$.

Prop. XXI.

(105.) *In general, in the single moveable pulley, the power is to the weight, as radius to twice the cosine of the angle which either string makes with the direction in which the weight acts.*

Let *AW* be the direction in which the weight acts; produce *BD* till it meets *AW* in *C*, from *A* draw *AD*



at right angles to *AC*, meeting *BC* in *D*; then if *CD*

be taken to represent the power at P, or the power which acts in the direction DB, CA will represent that part of it which is effective in sustaining the weight, and AD will be counteracted by an equal and opposite force, arising from the tension of the string CE; also, the two strings are equally effective in sustaining the weight; therefore $2AC$ will represent the whole weight sustained; consequently, $P : W :: CD : 2AC ::$ rad. : 2 cos. DCA.

(106.) Cor. 1. If the figure be inverted, and E and B be considered as a power and weight which sustain each other upon the fixed pulley A, W is the pressure upon the center of motion; consequently the power : the pressure :: radius : 2 cos. DCA.

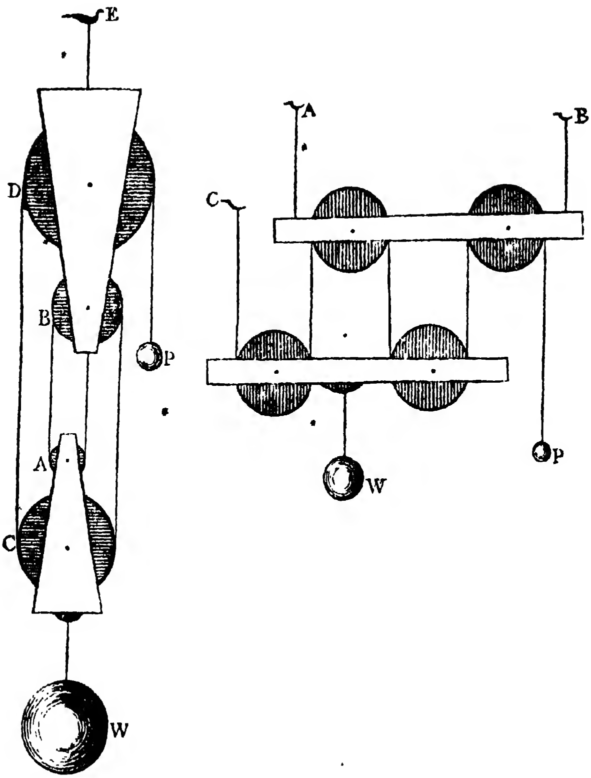
(107.) Cor. 2. When the strings are parallel, the angle DCA vanishes, and it's cosine becomes the radius; in this case, the power : the pressure :: 1 : 2.

Prop. XXII.

(108.) *In a system where the same string passes round any number of pulleys, and the parts of it between the pulleys are parallel,* P : W :: 1 : *the number of strings at the lower block.*

Since the parallel parts, or strings at the lower block, are in the direction in which the weight acts, they exactly support the whole weight; also, the tension in every point of these strings is the same, otherwise the system would not be at rest, and consequently each of them sustains an equal weight; whence

it follows that, if there be n strings, each sustains



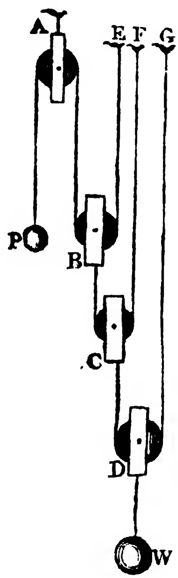
$\frac{1}{n}th$ part of the weight; therefore, P sustains $\frac{1}{n}th$ part of the weight, or $P : W :: \frac{1}{n} : 1 :: 1 : n$.

(109.) Cor. If two systems of this kind be combined, in which there are m and n strings, respectively, at the lower blocks, $P : W :: 1 : mn$.

Prop. XXIII.

(110.) *In a system where each pulley hangs by a separate string, and the strings are parallel,* P : W :: 1 : *that power of* 2 *whose index is the number of moveable pulleys.*

In this system, a string passes over the fixed pulley *A*, and under the moveable pulley *B*, and is fixed



at *E*; another string is fixed at *B*, passes under the moveable pulley *C*, and is fixed at *F*; &c. in such a manner that the strings are parallel.

Then, by Art. 104, when there is an equilibrium,

P : the weight at B :: 1 : 2
the weight at B : the weight at C :: 1 : 2
the weight at C : the weight at D :: 1 : 2
&c.

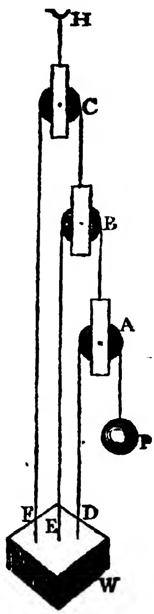
Comp. $P : W :: 1 : 2 \times 2 \times 2 \times$ &c. continued to as many factors as there are moveable pulleys; that is, when there are n such pulleys, $P : W :: 1 : 2^n$.

(111.) Cor. 1. The power and weight are wholly sustained at A, E, F, G, &c. which points sustain respectively, $2P$, P, $2P$, $4P$, &c.

(112.) Cor. 2. When the strings are not parallel, $P : W ::$ rad. : 2 cos. of the angle which the string makes with the direction in which the weight acts, in each case (Art. 105.)

Prop. XXIV.

(113.) *In a system of* n *pulleys each hanging by a separate string, where the strings are attached to the weight as is represented in the annexed figure,* $P : W :: 1 : 2^n - 1$.



string, fixed to the weight at F, passes over the

pulley C, and is again fixed to the pulley B; another string, fixed at E, passes over the pulley B, and is fixed to the pulley A; &c. in such a manner that the strings are parallel:

Then, if P be the power, the weight sustained by the string DA is P; also the pressure downwards upon A, or the weight which the string AB sustains, is $2P$ (Art. 107.); therefore the string EB sustains $2P$; &c. and the whole weight sustained is $P+2P+4P+$&c. Hence, $P : W :: 1 : 1+2+4+$&c. to n terms :: $1 : 2^n-1$ (*Alg.* Art. 222.).

(114.) Cor. 1. Both the power and the weight are sustained at H.

(115.) Cor. 2. When the strings are not parallel, the power in each case, is to the corresponding pressure upon the center of the pulley :: rad. : 2 cos. of the angle made by the string with the direction in which the weight acts (Art. 106.). Also, by the resolution of forces, the power in each case, or pressure upon the former pulley, is to the weight it sustains :: rad. : cos. of the angle made by the string with the direction in which the weight acts.

ON THE INCLINED PLANE.

Prop. XXV.

(116.) *If a body act upon a perfectly hard and smooth plane, the effect produced upon the plane is in a direction perpendicular to it's surface.*

Case 1. When the body acts perpendicularly upon the plane, it's force is wholly effective in that direction; since there is no cause to prevent the effect, or to alter it's direction.

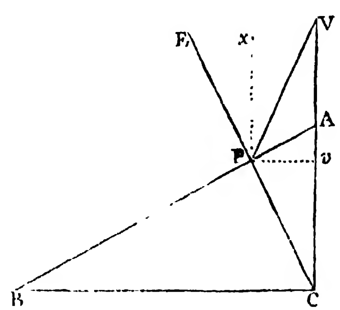
Case 2. When the direction in which the body acts is oblique to the plane, resolve it's force into two, one parallel, and the other perpendicular, to the plane; the former of these can produce no effect upon the plane, because there is nothing to oppose it in the direction in which it acts (see Art. 29.); and the latter is wholly effective (by the first case); that is, the effect produced by the force is in a direction perpendicular to the plane.

(117.) Cor. The reaction of the plane is in a direction perpendicular to it's surface (Art. 32.).

Prop. XXVI.

(118.) *When a body is sustained upon a plane which is inclined to the horizon,* P : W :: *the sine of the plane's inclination : the sine of the angle which the direction of the power makes with a perpendicular to the plane.*

Let *BC* be parallel to the horizon, *BA* a plane in-



clined to it; *P* a body, sustained at any point upon

the plane by a power acting in the direction PV. From P draw PC perpendicular to BA, meeting BC in C; and from C draw CV perpendicular to BC, meeting PV in V*. Then the body P is kept at rest by three forces which act upon it at the same time; the power, in the direction PV; gravity, in the direction VC; and the reaction of the plane, in the direction CP (Art. 117.); these three forces are therefore properly represented by the three lines PV, VC, and CP (Art. 56.); or $P : W :: PV : VC ::$ sin. PCV : sin. VPC; and in the similar triangles APC, ABC (Euc. 8. vi.), the angles ACP, and CBA are equal; therefore $P : W ::$ sin. ABC : sin. VPC.

(119.) Cor. 1. When PV coincides with PA, or the power acts parallel to the plane, $P : W :: PA : AC :: AC : AB$.

(120.) Cor. 2. When PV coincides with Pv, or the power acts parallel to the base, $P : W :: Pv : vC :: AC : CB$; because the triangles PvC, ABC are similar.

(121.) Cor. 3. When PV is parallel to CV, the power sustains the whole weight.

(122.) Cor. 4. Since $P : W ::$ sin. ABC : sin. VPC, by multiplying extremes and means, $P \times$ sin. $VPC = W \times$ sin. ABC; and if W, and the sine of the $\angle ABC$ be invariable, $P \propto \frac{1}{\text{sin. } VPC}$ (*Alg.* Art. 206.); therefore P is the least, when $\frac{1}{\text{sin. } VPC}$ is the least, or sin. VPC the greatest; that is, when sin. VPC becomes the radius, or PV coincides with PA. Also, P is indefinitely great when sin. VPC vanishes; that is, when the power acts perpendicularly to the plane.

* That PV, CV, are in the same plane, appears from Art. 55.

(123.) Cor. 5. If P and the $\angle ABC$ be given, $W \propto \sin. VPC$; therefore W will be the greatest when $\sin. VPC$ is the greatest, that is, when PV coincides with PA. Also, W vanishes when the $\sin. VPC$ vanishes, or PV coincides with PC.

(124.) Cor. 6. The power : the pressure :: $PV : PC :: \sin. PCV : \sin. PVC :: \sin. ABC : \sin. PVC$.

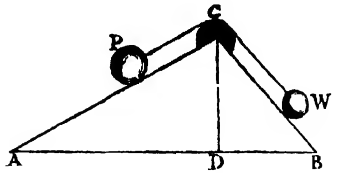
(125.) Cor. 7. When the power acts parallel to the plane, the power : the pressure :: $PA : PC :: AC : BC$.

(126.) Cor. 8. When the power acts parallel to the base, the power : the pressure :: $Pv : PC :: AC : AB$.

(127.) Cor. 9. $P \times \sin. PVC =$ the pressure $\times \sin. ABC$; and when P and the $\angle ABC$ are given, the pressure $\propto \sin. PVC$; therefore the pressure will be the greatest when PV is parallel to the base.

(128.) Cor. 10. When two sides of a triangle, taken in order, represent the quantities and directions of two forces which are sustained by a third, the remaining side, taken in the same order, will represent the quantity and direction of the third force (Art. 54.) Hence, if we suppose PV to revolve round P, when it falls between Px, which is parallel to VC, and PE, the direction of gravity remaining unaltered, the direction of the reaction must be changed, or the body must be supposed to be sustained against the under surface of the plane. When it falls between PE and xP produced, the direction of the power must be changed. And when it falls between xP produced, and PC, the directions of both the power and reaction must be different from what they were supposed to be in the proof of the proposition; that is, the body must be sustained against the under surface of the plane, by a force which acts in the direction VP.

(129.) Cor. 11. If the weights *P*, *W*, sustain each other upon the planes *AC*, *CB*, which have



a common altitude *CD*, by means of a string *PCW* which passes over the pulley *C*, and is parallel to the planes, then $P : W :: AC : BC$.

For, since the tension of the string is every where the same, the sustaining power, in each case, is the same; and calling this power x,

$$P : x :: AC : CD \text{ (Art. 119.)};$$
$$x : W :: CD : CB.$$
$$\text{comp. } P : W :: AC : CB.$$

ON THE WEDGE.

(130.) Def. A *Wedge* is a triangular prism; or a solid generated by the motion of a plane triangle parallel to itself, upon a straight line which passes through one of it's angular points*.

* See also Euc. B. XI. Def. 13.

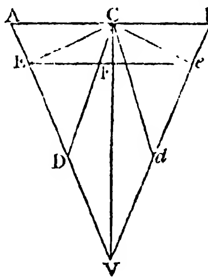
Knives, swords, coulters, nails, &c. are instruments of this kind.

The wedge is called *isosceles* or *scalene*, according as the section of it, made by a plane perpendicular to it's sides, is an *isosceles* or *scalene* triangle.

Prop. XXVII.

(131.) *If two equal forces act upon the sides of an isosceles wedge at equal angles of inclination, and a force act perpendicularly upon the back, they will keep the wedge at rest, when the force upon the back is to the sum of the forces upon the sides, as the product of the sine of half the vertical angle of the wedge and the sine of the angle at which the directions of the forces are inclined to the sides, to the square of radius.*

Let AVB represent a section of the wedge, made by a plane perpendicular to it's sides; draw VC per-



pendicular to AB; DC, dC, in the directions of the forces upon the sides; and CE, Ce, at right angles to AV, BV; join Ee, meeting CV in F.

Then, in the triangles VCA, VCB, since the angles VCA, CAV, are respectively equal to VCB, VBC, and VC is common to both, $AC = CB$, and the $\angle CVA = \angle CVB$. Again, in the triangles ACD, BCd,

the angles DAC, CDA, are equal to the angles CBd, BdC, and $AC = BC$; therefore, $DC = dC$. In the same manner it may be shewn that $CE = Ce$, and $AE = Be$; hence the sides AV, BV, of the triangle AVB, are cut proportionally in E and e; therefore Ee is parallel to AB (EUC. 2. vi), or perpendicular to CV; also, since $CE = Ce$, and CF is common to the right-angled triangles CEF, CeF, we have $EF = eF$ (EUC. 47. i.)

Now since DC and dC are equal, and in the directions of the forces upon the sides, they will represent them; resolve DC into two, DE, EC, of which DE produces no effect upon the wedge, and EC, which is effective (Art. 116.), does not wholly oppose the power, or force upon the back; resolve EC therefore into two, EF, parallel to the back, and FC perpendicular to it, the latter of which is the only force which opposes the power. In the same manner it appears that eF, FC, are the only effective parts of dC, of which FC opposes the power, and eF is counteracted by the equal and opposite force EF; hence if $2CF$ represent the power, the wedge will be kept at rest*; that is, when the force upon the back : the sum of the resistances upon the sides :: $2CF$: $DC + dC$ :: $2CF$ $2DC$ :: $CF : DC$; and

$$CF : CE \quad \text{sin. } CEF : \text{rad.} :: \text{sin. } CVE : \text{rad.}$$
$$CE : DC :: \text{sin. } CDE : \text{rad.}$$

Comp. $CF : DC :: \text{sin. } CVE \times \text{sin. } CDE : \overline{\text{rad.}}|^2$

(132.) COR. 1. The forces do not sustain *each other*, because the parts DE, de, are not counteracted.

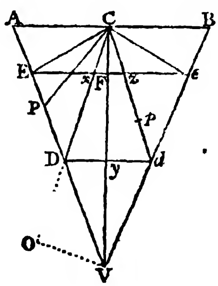
* The directions of the three forces must meet in a point, otherwise a rotatory motion will be given to the wedge.

(133.) Cor. 2. If the resistances act perpendicularly upon the sides of the wedge, the angle CDE becomes a right angle, and P : the sum of the resistances :: sin. $CVE \times$ rad. : $\overline{\text{rad.}}|^2$ :: sin. CVE : rad. :: $AC : AV$.

(134.) Cor. 3. If the directions of the resistances be perpendicular to the back, the angle $CDE = \angle CVE$, and P : the sum of the resistances : $\overline{\text{sin. } CVE}|^2 : \overline{\text{rad.}}|^2$ :: $AC^2 : AV^2$.

(135.) Cor. 4. When the resistances act parallel to the back, sin. $CDA =$ sin. CAV, and P : the sum of the resistances :: sin. $CVA \times$ sin. CAV : $\overline{\text{rad.}}|^2$ :: $CA \times CV : AV^2$ :: $CE \times AV$* : AV^2 :: $CE : AV$.

(136.) Cor. 5. In the demonstration of the proposition it has been supposed that the sides of the wedge are perfectly smooth; if on account of the friction, or by any other means, the resistances are *wholly* effective, join Dd, which will cut CV at right angles



in y, and resolve DC, dC into Dy, yC, dy, yC, of which Dy and dy destroy each other, and $2yC$ sustains the power. Hence, the power : the sum of the resistances :: $2yC : 2DC$ :: $yC : DC$ :: sin. CDy or DCA : rad.

* By similar triangles, $CE : CA$ $CV : AV$; therefore $CE \times AV = CA \times CV$.

(137.) Cor. 6. If Ee cut DC and dC in x and x, the forces, xC, xC, when wholly effective, and the forces DC, dC, acting upon smooth surfaces, will sustain the same power $2CF$.

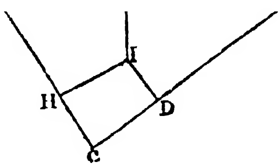
(138.) Cor. 7. If from any point P in the side AV, PC be drawn, and the resistance upon the side be represented by it, the effect upon the wedge will be the same as before; the only difference will be in the part PE which is ineffective.

(139.) Cor. 8. If DC be taken to represent the resistance on one side, and pC, greater or less than dC, represent the resistance on the other, the wedge cannot be kept at rest by a power acting upon the back; because, on this supposition, the forces which are parallel to the back are unequal.

This Proposition and it's Corollaries have been deduced from the actual resolution of the forces, for the purpose of shewing what parts are lost, or destroyed by their opposition to each other; the same conclusions may, however, be very concisely and easily obtained from Art. 142.

Prop. XXVIII.

(140.) *When three forces, acting perpendicularly upon the sides of a scalene wedge, keep each other in equilibrio, they are proportional to those sides.*



Let GI, HI, DI, the directions of the forces,

meet in I; then since the forces keep each other at rest, they are proportional to the three sides of a triangle which are respectively perpendicular to those directions (Art. 59.); that is, to the three sides of the wedge.

(141.) Cor. 1. If the lines of direction, passing through the points of impact, do not meet in a point, the wedge will have a rotatory motion communicated to it; and this motion will be round the center of gravity of the wedge. (See Art. 184.)

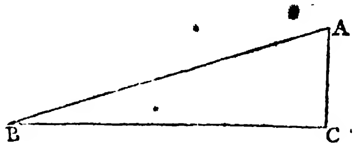
(142.) Cor. 2. When the directions of the forces are not perpendicular to the sides, the effective parts must be found, and there will be an equilibrium when those parts are to each other as the sides of the wedge.

ON THE SCREW.

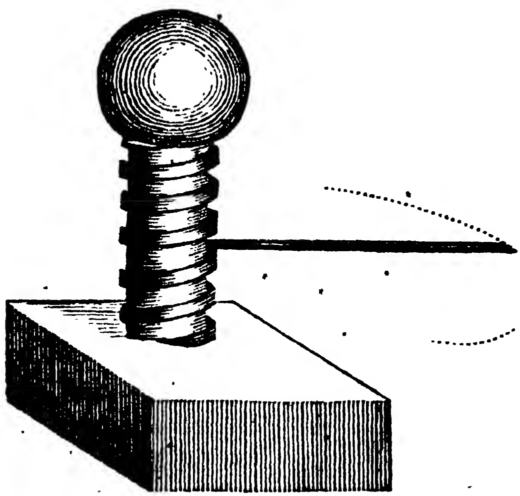
(143.) Def. The *Screw* is a mechanical power, which may be conceived to be generated in the following manner:

Let a solid and a hollow cylinder of equal diameters be taken, and let ABC be a right-angled plane triangle whose base BC is equal to the circumference of the solid cylinder; apply the triangle to the convex surface of this cylinder, in such a manner, that the base BC may coincide with the circumference of the base of the cylinder, and BA will form a spiral thread on it's surface. By applying to the cylinder, triangles, in succession, similar and equal to ABC, in such a manner, that their bases may be parallel to BC, the spiral thread may be continued; and supposing this thread to

have thickness, or the cylinder to be protuberant where it falls, the external screw will be formed, in which the



distance between two contiguous threads, measured in a direction parallel to the axis of the cylinder, is *AC*. Again, let the triangles be applied in the same manner to the concave surface of the hollow cylinder, and where the thread falls let a groove be made, and the internal screw will be formed. The two screws being thus exactly adapted to each other, the solid or hollow cylinder, as the case requires, may be moved round

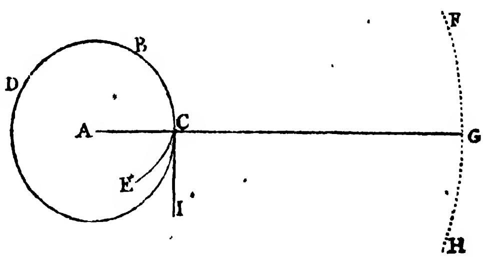


the common axis, by a lever perpendicular to that axis; and a motion will be produced in the direction of the axis, by means of the spiral thread.

Prop. XXIX.

(144.) *When there is an equilibrium upon the screw,* P : W :: *the distance between two contiguous threads, measured in a direction parallel to the axis* : *the circumference of the circle which the power describes.*

Let *BCD* represent a section of the screw made by a plane perpendicular to it's axis, *CE* a part of the spiral thread upon which the weight is sustained; then *CE* is a portion of an inclined plane, whose height is the distance between two threads, and base equal to the circumference *BCD*. Call *F* the power which acting at *C* in the plane *BCD*, and in the direction *CI* perpendicular to *AC*, will sustain the weight *W*, or prevent the motion of the screw round the axis; then since the weight is sustained upon the inclined plane



CE by a power *F* acting parallel to it's base, *F* : *W* :: the height : the base (Art. 120.) :: the distance between two threads : the circumference *BCD*. Now, instead of supposing the power *F* to act at *C*, let a power *P* act perpendicularly at *G*, on the straight lever *GCA*, whose center of motion is *A*, and let this power produce the same effect at *C* that *F* does; then, by the property of

the lever, $P : F :: CA : GA$: the circumference BCD : the circumference FGH. We have therefore these two proportions,

$$F : W :: \text{distance between two threads} : BCD$$
$$P : F :: \quad BCD \quad : FGH$$
$$\text{comp.} \cdot P : W :: \text{distance between two threads} : FGH.$$

(145.) Cor. 1. In the proof of this Proposition the whole weight is supposed to be sustained at one point C of the spiral thread; if we suppose it to be dispersed over the whole thread, then, by the Proposition, the power at G necessary to sustain any part of the weight : that part :: the distance between two threads : the circumference of the circle FGH; therefore the sum of all these powers, or the whole power : the sum of all the corresponding weights, or the whole weight, :: the distance between two threads : the circumference of the circle FGH (*Alg.* Art. 183.).

(146.) Cor. 2. Since the power, necessary to sustain a given weight, depends upon the distance between two threads and the circumference FGH, if these remain unaltered, the power is the same, whether the weight is supposed to be sustained at C, or at a point upon the thread nearer to, or farther from, the axis of the cylinder.

(147.) Some Authors have deduced the properties of the mechanical powers *immediately* from the Third Law of Motion, contending that if the power and weight be such as would sustain each other, and the machine be put into motion, the momenta of the power and weight are equal; and consequently, that the power $\times$ the velocity of the power $=$ the weight

× the velocity of the weight; or *the power's velocity* : *the weight's velocity* :: *the weight* : *the power.*

Though this conclusion be just, the reasoning by which it is attempted to be proved is inadmissible, because the Third Law of Motion relates to the action of one body immediately upon another (Art. 36.). It may however be deduced from the foregoing Propositions; and as it is, in many cases, the simplest method of estimating the power of a machine, it may not be improper to establish it's truth.

In the application of the rule, two things must be attended to: 1st, We must estimate the velocity of the power or weight in the direction in which it acts. 2dly, We must estimate that part only of the power or weight which is effective.

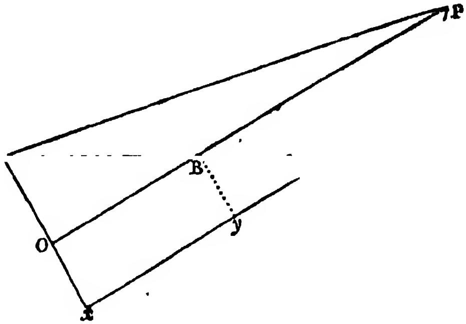
These two considerations are suggested by the Second Law of Motion, according to which motion is communicated in the *direction* of the force *impressed,* and is proportional to that force.

Prop. XXX.

(148.) *The velocity of a body in any one direction* AB *being given, to estimate it's velocity in any other direction* BP.

Suppose the motion of *A* to be produced by a force acting in the direction *BP,* by means of a string which passes over a pulley at *P*; produce *PB* to *O,* making *PO* = *PA*; join *AO*; then *OB* is the space which measures the approach of *A* to *P*. Now let the pulley be removed to such a distance that the angle at *P* may be considered as evanescent, and the power will always act in

the same direction BP; also, the angles at A and O are equal; and they are right angles, because the three

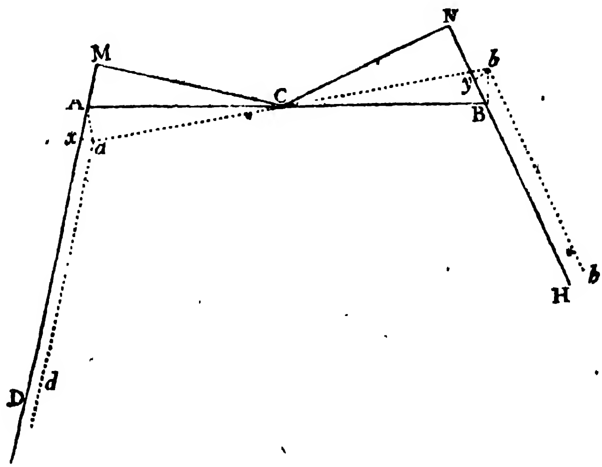


angles of the triangle APO are equal to two right angles, and the angle at P vanishes; therefore, the space described in the direction OP, or BP, is determined by drawing AO perpendicular to OP. If the space described in the direction xy, which is parallel to OP, be required, produce AO to x, and from B draw By at right angles to xy; then the figure $OByx$ is a parallelogram, and $OB = xy$ the space required. Also, if the motion in the direction AB be uniform, the motion in the direction BP, or xy, is uniform; since $AB : OB ::$ rad. : cos. ABO. Hence, the velocity in the direction AB : the velocity in the direction $BP :: AB : OB$ (Art. 11.).

Prop. XXXI.

(149.) *If a power and weight sustain each other on any machine, and the whole be put in motion, the velocity of the power : the velocity of the weight :: the weight : the power.*

CASE 1. *In the lever ACB*, let a power and weight, acting in the directions *AD*, *BH*, sustain each other, and let the machine be moved uniformly round the



center *C*, through a very small angle *ACa*; Join *Aa*, *Bb*; draw *CM*, *ax*, at right angles to *MD*; and *CN*, *by*, at right angles to *NB*; then *A*'s velocity : *B*'s velocity :: *Ax* : *By* (Art. 148.). Now the triangles *Axa*, *MCA*, are similar; because $\angle xAC = \angle AMC + \angle MCA$ (Euc. 32. i.) and $\angle aAC = \angle AMC$; therefore, $\angle xAa = \angle MCA$; and the angles at *M* and *x* are right angles; consequently, the remaining angles are equal; and

$$Ax : Aa :: CM : CA;$$

also, in the sim. △ˢ *ACa*, *BCb*, $Aa : Bb :: CA : CB$; and in the sim. △ˢ *Bby*, *BCN*, $Bb : By :: CB : CN$; by composition, $Ax : By :: CM : CN ::$ the weight : the power (Art. 81.); or the power's velocity : the weight's velocity :: the weight : the power.

Case 2. *In the wheel and axle*, if the power be made to describe a space equal to the circumference of the wheel with an uniform motion, the weight will be uniformly raised through a space equal to the circumference of the axle; hence, the power's velocity : the weight's velocity :: the circumference of the wheel : the circumference of the axle :: the radius of the wheel : the radius of the axle :: the weight : the power (Art. 96.).

Case 3. *In the single fixed pulley*, if the weight be uniformly raised 1 inch, the power will uniformly describe 1 inch in the direction of it's action; therefore the power's velocity : the weight's velocity :: the weight : the power.

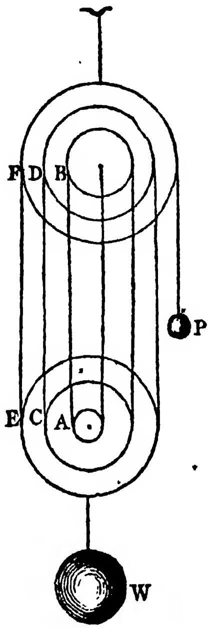
Case 4. *In the single moveable pulley where the strings are parallel*, if the weight be raised 1 inch, each of the strings is shortened 1 inch, and the power describes 2 inches; therefore, *P*'s velocity : *W*'s velocity :: *W* : *P* (Art. 104.).

Case 5. *In the system of pulleys described in Art.* 108, if the weight be raised 1 inch, each of the strings at the lower block is shortened 1 inch, and the power describes *n* inches; therefore, *P*'s velocity : *W*'s velocity :: *W* : *P*.

In this system of pulleys, whilst 1 inch of the string passes over the pulley *A*, 2 inches pass over the pulley *B*, 3 over *C*, 4 over *D*, &c.

Hence it follows, that if in the solid block *A*, the grooves *A*, *C*, *E*, &c. be cut, whose radii are 1, 3, 5, &c. and in the block *B*, the grooves *B*, *D*, *F*, &c. whose radii are 2, 4, 6, &c. and a string be passed round these grooves as in the annexed figure; the grooves will answer the purpose of so many distinct pulleys,

and every point in each, moving with the velocity of the string in contact with it, the whole friction will be



removed to the two centers of motion in the blocks *A* and *B*.

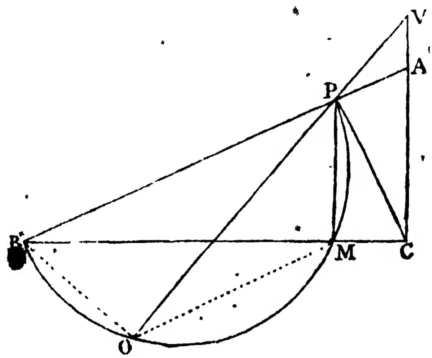
CASE 6. *In the system of pulleys described in Art.* 110, each succeeding pulley moves twice as fast as the preceding;

therefore, *W*'s velocity : *C*'s velocity :: 1 : 2
C's velocity : *B*'s velocity :: 1 : 2
B's velocity : *P*'s velocity :: 1 : 2
&c.

comp. *W*'s velocity : *P*'s velocity :: 1 : 2×2×2 ×&c. :: *P* : *W*.

CASE 7. *In the system, Art.* 113, if the weight be raised 1 inch, the pulley B will descend 1 inch, and the pulley A will descend $2+1$ inches; in the same manner, the next pulley will descend $2 \times \overline{2+1}+1$ inches, or $4+2+1$ inches; &c. therefore P's velocity : W's velocity :: $1+2+4+$&c. : 1 :: W : P.

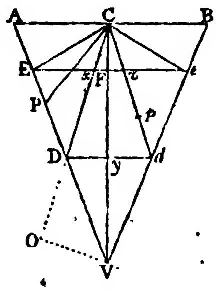
CASE 8. Let a body be uniformly raised along *the inclined plane BA* from B to P, by a power acting parallel to PV; upon BP describe a semi-circle BOP,



cutting BC in M; produce VP to O, join BO, PM, MO. Then since the angles BOP, BMP, in the semi-circle, are right angles, OP and MP are spaces uniformly described in the same time, by the power and weight in their respective directions (Art. 148.); also, because $\angle POM = \angle PBM = \angle PCV$, and $\angle OPM = \angle PVC$ (Euc. 29. i.), the triangles POM PVC are similar, and $OP : MP :: VC : PV$, or the power's velocity : the weight's velocity :: the weight : the power, in the case of an equilibrium (Art. 118.)

CASE 9. *In the isosceles wedge, xC* is the only effective part of the resistance DC (see Art. 137.);

draw VO perpendicular to CD produced; then if the wedge be moved uniformly from C to V, CO is



the space uniformly described by the resisting force (Art. 148.); hence, the power's velocity : the velocity of the resisting force :: CV : CO :: Cx : CF :: the resistance : the power.

Case 10. *In the screw,* whilst the power uniformly describes the circumference of the circle FGH (Art. 144.), the weight is uniformly raised through the distance between two contiguous threads; therefore P's velocity : W's velocity :: the circumference of the circle FGH : the distance between two threads :: W : P.

Case 11. *In any combination of the mechanical powers,* let $P : W$, $W : R$, $R : S$, &c. be the ratios of the power and weight in each case, when there is an equilibrium; then,

P's velocity : W's velocity :: W : P
W's velocity : R's velocity :: R : W
R's velocity : S's velocity :: S : R
&c.
comp. P's velocity : S's velocity :: S : P.

SCHOLIUM.

(150.) It has been usual to distinguish Levers into three kinds, according to the different situations of the power, weight, and center of motion; there are however only two kinds which essentially differ; those in which the forces act on *contrary* sides of the center of motion, as the common balance, steel-yard, &c. and those in which they act on the *same* side, as the stock-knife, shears which act by a spring, oars, &c. The proportion between the forces, when there is an equilibrium, is expressed in the same terms in each case; but the levers differ in this respect, that the pressure upon the fulcrum depends upon the sum of the forces in the former case, and upon their difference in the latter; and consequently, the friction upon the center of motion, *cæteris paribus*, is greater in the former case than the latter.

(151.) The pulley has, by some Writers, been referred to the lever, and they have justly deduced it's properties from the proportions which are found to obtain in that mechanical power; for, the adhesion of the pulley and the rope, which takes place at the circumference of the pulley, will overcome the friction at the center of motion; both because it acts at a mechanical advantage, and because the surface in contact is greater in the former case than in the latter; and the friction depends, not only upon the weight sustained, but also upon the quantity of surface in contact: Thus, in practice, the rope and pulley move on together, and the pulley becomes a lever.

(152.) The Wedge has hitherto chiefly been applied to the purposes of separating the parts of bodies, and it's power, notwithstanding the friction, is much greater than the theory leads us to expect; the reason is, the effect is produced by impact, and the momentum thus generated is incomparably greater than the effect of pressure, in the same time. Mr. ECKHARD, a very ingenious mechanic, by combining it with the wheel and axle, has constructed a machine, the power of which, considering it's simplicity, is much greater than that of any machine before invented.

SECTION V.

ON THE CENTER OF GRAVITY.

(153.) Def. The *Center of Gravity* of any body, or system of bodies, is that point upon which the body or system, acted upon only by the force of gravity, will balance itself in all positions*.

(154.) Hence it follows, that if a line or plane, which passes through the center of gravity, be supported, the body, or system, will be supported in all positions.

(155.) Conversely, if a body, or system, balance itself upon a line or plane, in all positions, the center of gravity is in that line or plane.

If not, let the line or plane be moved parallel to itself till it passes through the center of gravity, then we have increased both the quantity of matter on one side of the line or plane, and it's distance from the line or plane, and diminished both, on the other side; hence, if the body balanced itself in all positions in the former case, it cannot, from the nature of the

* That there is such a point in every body, or system of bodies, will be shewn hereafter.

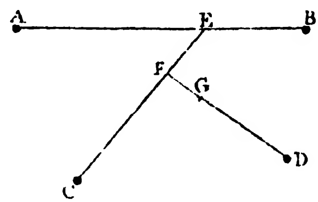
lever, balance itself in *all positions*, in the latter; consequently, the center of gravity is not in this line, or plane (Art. 154.), which is contrary to the supposition.

(156.) Cor. By reasoning in the same manner, it appears that a body, or system of bodies, cannot have more than one center of gravity.

Prop. XXXII.

(157.) *To find the center of gravity of any number of particles of matter.*

Let A, B, C, D, &c. be the particles; and suppose A, B, connected by the inflexible line AB without weight*; divide AB into two parts in E, so that



$A : B :: BE : EA$, or comp. $A+B : B :: AB : EA$; then will A and B balance each other upon E, or if E be supported, A and B will be supported in all positions (Art. 86.); let E be supported on the line CE, then are A and B supported in all positions; also the pressure upon the point E is equal to the sum of the weights A and B (Art. 70. Ax. 1.) Join EC, and take $A+B : C :: CF : FE$, or $A+B+C : C :: EC : FE$; then if F be supported, E and C will be supported, that is, A, B, and C, will be supported, in all

* The particles must be supposed to be connected, otherwise they could not act upon each other, so as to balance upon any point.

positions of the system; and the pressure upon F will be the sum of the weights, A, B, and C. In the same manner, join FD, and divide it into two parts in G, so that $A+B+C : D :: DG : FG$, or $A+B+C+D : D :: FD : FG$, and the system will balance itself in all positions upon G; that is, G is the center of gravity of the system.

(158.) COR. 1. From this Proposition it appears that every body, or system of bodies, has a center of gravity.

(159.) COR. 2. If the particles be supposed to be connected in any other manner, the same point G will be found to be their center of gravity (Art. 156.)

(160.) COR. 3. The effect of any number of particles in a system, to produce or destroy an equilibrium, is the same, whether they are dispersed, or collected in their common center of gravity.

(161.) COR. 4. If A, B, C, &c. be bodies of finite magnitudes, G, the center of gravity of the system, may be found by supposing the bodies collected in their respective centers of gravity.

(162.) COR. 5. If the bodies A, B, C, &c. be increased or diminished in a given ratio, the same point G will be the center of gravity of the system. For the points E, F, G, depend upon the relative, and not upon the absolute weights of the bodies.

(163.) COR. 6. If any forces, which are proportional to the weights, act in parallel directions at A, B, C, D, they will sustain each other upon the point G; and this point is still called the center of gravity, though the particles are not acted upon by the force of gravity.

(164.) COR. 7. A force applied at the center of

gravity of a body cannot produce a rotatory motion in it. For every particle resists, by it's inertia, the communication of motion, and in a direction opposite to that in which the force applied tends to communicate the motion; these resistances, therefore, of the particles, act in parallel directions, and they are proportional to the weights (Art. 25.); consequently, they will balance each other upon the center of gravity.

Prop. XXXIII.

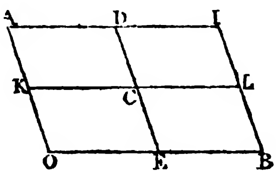
(165.) *To find the center of gravity of a right line*.*

The center of gravity of a right line, composed of particles of matter which are equal to each other and uniformly dispersed, is it's middle point. For, there are equal weights on each side, equally distant from the middle point, which will sustain each other, in all positions, upon that point (Art. 86.)

Prop. XXXIV.

(166.) *To find the center of gravity of a parallelogram.*

Let AB be an uniform lamina of matter in the



form of a parallelogram; bisect AO, AI, in K and D;

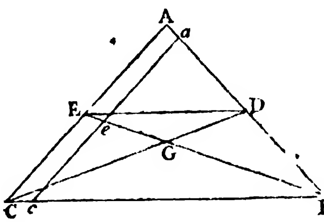
* When we speak of a line or plane as having a center of gravity, we suppose it to be made up of particles of matter, uniformly diffused over it.

draw KL, DE, respectively, parallel to AI, AO, cutting each other in C; this point C is the center of gravity of the figure. For if the parallelogram be supposed to be made up of lines parallel to AI, any one of these, as KL, is bisected by the line DE (since AC, CI, are parallelograms, and therefore, $KC = AD = DI = CL$); consequently, each line will balance itself upon DE (Art. 165.), or the whole figure will balance itself upon DE, in all positions; therefore, the center of gravity is in that line (Art. 155.) In the same manner it may be shewn that the center of gravity of the figure is in the line KL, consequently C, the intersection of the two lines, is the center of gravity required.

Prop. XXXV.

(167.) *To find the center of gravity of a triangle.*

Let ABC be an uniform lamina of matter in the form of a triangle; bisect AB, AC, in D, E; join



CD, BE, cutting each other in G, this point is the center of gravity of the triangle.

Suppose the triangle to be made up of lines parallel to CA, of which let cea be one; then since the triangles BEC, Bec, are similar,

$BE : EC :: Be : ec$; also, in the triangles BEA, Bea, $EA : BE :: ea : Be$;
by composition, $EA : EC :: ea : ec$; and $EA = EC$, therefore $ea = ec$; and consequently, the line ac will balance itself in all positions upon BE. For the same reason, every other line parallel to AC will balance itself, in all positions, upon BE, or the whole triangle will balance itself in all positions upon BE; therefore the center of gravity of the triangle is in that line. In the same manner it may be proved that the center of gravity is in the line CD; therefore it is in G, the intersection of the two lines BE, CD.

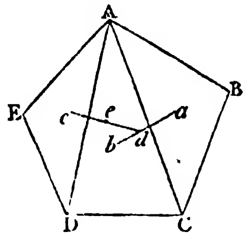
(168.) Cor. The distance of G from B is two-thirds of the line BE. Join ED; then since $AD = DB$, and $AE = EC$, ED is parallel to BC (Euc. 2. vi.); therefore, the triangles AED, ACB, are similar, and $CB : CA :: ED : EA$; alternately, $CB : ED :: CA : EA :: 2 : 1$. Also, the triangles CGB, EGD, are similar, therefore, $BG : CB :: GE : ED$; alternately, $BG : GE :: CB : ED :: 2 : 1$; hence, $BG : BE :: 2 : 3$.

Prop. XXXVI.

(169.) *To find the center of gravity of any rectilinear figure.*

Let $ABCDE$ be an uniform lamina of matter of the proposed figure. Divide it into the triangles ABC, ACD, ADE, whose centers of gravity a, b, c, may be found by the last Proposition; then if the triangles be collected in their respective centers of gravity (Art. 160.), their common center of gravity may be found as in Prop. 32.; that is, join ab and take

db : *ad* :: the triangle *ABC* : the triangle *ADC*, and *d* is the center of gravity of the two triangles



ABC, *ACD*. Join *dc*, and take *ce* : *ed* :: the sum of the triangles *ABC*, *ACD* : the triangle *AED*, and *e* is the center of gravity of the figure.

Prop. XXXVII.

(170.) *To find the center of gravity of any number of bodies placed in a straight line.*

Let *A*, *B*, *C*, *D*, be the bodies, collected in their respective centers of gravity; *S* any point in the straight line *SAD*; *O* the center of gravity of all the bodies. Then since the bodies balance each other

upon *O*, $A\times AO+B\times BO=C\times CO+D\times DO$ (See Art. 92.); that is, $A\times\overline{SO-SA}+B\times\overline{SO-SB}$ $=C\times\overline{SC-SO}+D\times\overline{SD-SO}$; hence, by mult. and transposition, $A\times SO+B\times SO+C\times SO+D\times$

$SO = A \times SA + B \times SB + C \times SC + D \times SD$; therefore,

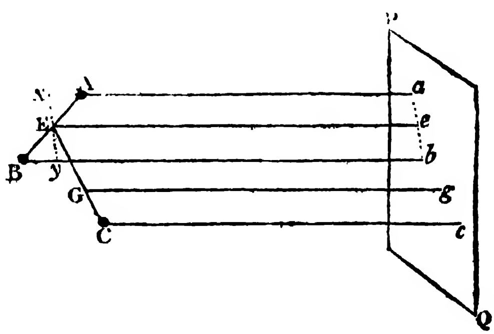
$$SO = \frac{A \times SA + B \times SB + C \times SC + D \times SD}{A + B + C + D}.$$

(171.) Cor. If any of the bodies lie the other way from S, their distances must be reckoned negative; and if SO be negative, the distance SO must be measured from S in that direction which, in the calculation, was supposed to be negative. (See *Alg.* Art. 472.)

Prop. XXXVIII.

(172.) *If perpendiculars be drawn from any number of bodies to a given plane, the sum of the products of each body, multiplied by it's perpendicular distance from the plane, is equal to the product of the sum of all the bodies multiplied by the perpendicular distance of their common center of gravity from the plane.*

Let A, B, C, &c. be the bodies, collected in their respective centers of gravity; PQ the given



plane; draw Aa, Bb, Cc, at right angles to PQ, and

consequently, parallel to each other (Euc. 6. xi.); join AB, and take $AE : EB :: B : A$, then E is the center of gravity of A and B; through E draw Ee perpendicular to PQ, or parallel to Aa, and xEy perpendicular to Aa or Bb; then in the similar triangles AEx, EBy, $Ax : AE :: By : BE$, alternately, $Ax : By :: AE : BE :: B : A$; therefore $A \times Ax = B \times By$, that is, $A \times \overline{xa - Aa} = B \times \overline{Bb - yb}$, or since Ea, Eb, are parallelograms, $A \times \overline{Ee - Aa} = B \times \overline{Bb - Ee}$; and by multiplication and transposition, $A \times Ee + B \times Ee = A \times Aa + B \times Bb$, that is, $\overline{A + B} \times Ee = A \times Aa + B \times Bb$.

Again, join EC, and take $CG : GE :: A + B : C$, then G is the center of gravity of the bodies A, B, C; draw Gg perpendicular to PQ; and it may be shewn, as before, that $\overline{A + B} \times Ee + C \times Cc = \overline{A + B + C} \times Gg$, or $A \times Aa + B \times Bb + C \times Cc = \overline{A + B + C} \times Gg$. The same demonstration may be extended to any number of bodies.

(173.) Cor 1. Hence $Gg = \dfrac{A \times Aa + B \times Bb + C \times Cc}{A + B + C}$; and if a plane be drawn parallel to PQ, and at the distance Gg from it, the center of gravity of the system lies somewhere in this plane. In the same manner two other planes may be found, in each of which the center of gravity lies, and the point where the three planes cut each other, is the center of gravity of the system.

(174.) Cor. 2. If any of the bodies lie on the other side of the plane, their distances must be reckoned negative.

(175.) Cor. 3. Wherever the bodies are situated,

if their respective perpendicular distances from the plane remain the same, the distance of their common center of gravity from the plane will remain the same.

(176.) Cor. 4. Let the bodies lie in the same plane, and let perpendiculars, Aa, Bb, Cc, Gg, be drawn to any given *line* in that plane, then

$$Gg = \frac{A \times Aa + B \times Bb + C \times Cc}{A + B + C}.$$

(177.) Cor. 5. If A and B be on one side of the plane, and C on the other, and the plane pass through the center of gravity, then $A \times Aa + B \times Bb = C \times Cc$. For $Gg \times \overline{A + B + C} = A \times Aa + B \times Bb - C \times Cc$, and $Gg = 0$, therefore $A \times Aa + B \times Bb - C \times Cc = 0$; or $A \times Aa + B \times Bb = C \times Cc$.

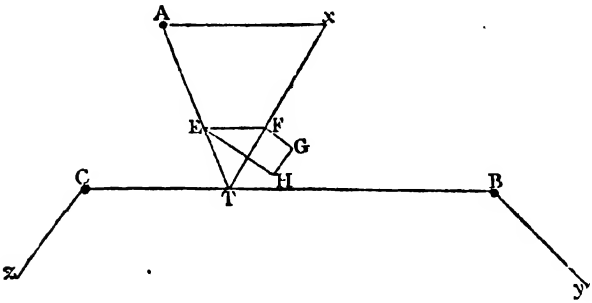
Prop. XXXIX.

(178.) *If any momenta be communicated to the parts of a system, it's center of gravity will move in the same manner that a body, equal to the sum of the bodies in the system, would move, were it placed in that center, and the same momenta, in the same directions, communicated to it.*

Let A, B, C, be the bodies in the system, and the points A, B, C, their respective centers of gravity; join BC, and take $BT : TC :: C : B$; join AT, and take $TE : EA :: A : B + C$, or $TE : TA :: A : A + B + C$, then will E be the center of gravity of the system (Art. 161.)

Suppose the momentum communicated to A would cause it to move from A to x in T'', and at x let the body be stopped; join Tx, and take $TF : Tx :: A : A + B + C$, then F is the center of gravity of the bodies

when they are at x, B, C; join EF, and since $TE : TA :: A : A+B+C :: TF : Tx$, EF is parallel to Ax (Euc. 2. vi.), and consequently the triangles TEF, TAx, are similar; therefore $EF : Ax :: A : A+B+C$.



Hence if one body A in the system be moved from A to x, the center of gravity is moved from E to F; which point may be thus determined; draw EF parallel to Ax, and take $EF : Ax :: A : A+B+C$.

Next let a momentum be communicated to B, which would cause it to move from B to y in T'''; at y let the body be stopped; then, according to the rule above laid down, draw FG parallel to By, and take $FG : By :: B : A+B+C$, and G will be the center of gravity of the bodies when they are at x, y, C. In the same manner, let a momentum be communicated to C, which would cause it to move from C to z in T'', and at z let the body be stopped; draw GH parallel to Cz, and take $GH : Cz :: C : A+B+C$, then H is the center of gravity of the bodies when they are at x, y, z. If now the momenta, instead of being communicated separately, be communicated *at the same instant* to the bodies, at the end of T'' they will be

found in x, y, z, respectively; therefore, at the end of T'', their common center of gravity will be in H.

Now let E be a body equal to $A+B+C$, and let the same momentum be communicated to it that was before communicated to A, and in the same direction; then since EF is parallel to Ax, EF is in the *direction* in which the body E will move; also, since the quantities of motion communicated to A and E are equal, their velocities are reciprocally proportional to their quantities of matter (Art. 19.), or E's velocity : A's velocity :: $A : A+B+C$ :: $EF : Ax$; therefore, EF and Ax are spaces described by E and A in equal times (Art. 11.), or E will describe the space EF in T''. In the same manner FG is the space which the body E will describe in T'', if the momentum, before communicated to B, be communicated to it; and GH the space it will describe in T'', if the momentum before communicated to C, be communicated to it; join EH; and when the motions are communicated *at the same instant* to E, it will describe EH in T'' (Art. 42.) Hence it follows, that when the same momenta are communicated to the parts of a system, and to a body, equal to the sum of the bodies, placed in the common center of gravity, this body and the center of gravity are in the same point at the end of T''; and T may represent any time; therefore, they are always in the same point.

The same demonstration may be applied, whatever be the number of bodies in the system.

(179.) Cor. 1. If the parts of a system move uniformly in right lines, the center of gravity will either remain at rest, or move uniformly in a right line. For

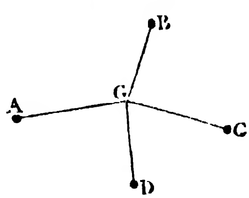
if the momenta communicated to the several parts of the system be communicated to a body, equal to the sum of the bodies, placed in the center of gravity of the system, it will either remain at rest or move uniformly in a right line (Art. 27.)

(180.) Cor. 2. If two weights support each other upon any machine, and it be put in motion, the center of gravity of the weights will neither ascend nor descend. For the momenta of the weights, in a direction perpendicular to the horizon, are equal and opposite (Art. 149.); therefore, if they were communicated to a body equal to the sum of the bodies, placed in the common center of gravity, they would neither cause it to ascend nor descend.

(181.) Cor. 3. The motion or quiescence of the center of gravity is not affected by the mutual action of the parts of a system upon each other. For action and reaction are equal and in opposite directions, and equal and opposite momenta communicated to a body, equal to the sum of the bodies in the system, will not disturb it's motion or quiescence.

(182.) Cor. 4. The effect of any force to communicate motion to the common center of gravity, is the same, upon whatever body in the system it acts.

(183.) Cor. 5. If G be the center of gravity of the



particles of matter, A, B, C, D, which are acted

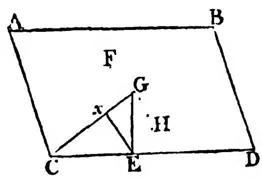
upon only by their mutual attractions, they will meet at *G*. For they must meet, and their common center of gravity will remain at rest (Art. 181.); therefore, they must meet at that center.

(184.) Cor. 6. If a rotatory motion be communicated to a body, and it be then left to move freely, the axis of rotation will pass through the center of gravity. For the center of gravity itself, either remaining at rest or moving uniformly forward in a right line, has no rotation.

Prop. XL.

(185.) *If a body be placed upon an horizontal plane, and a line drawn from it's center of gravity perpendicular to that plane, the body will be sustained, or not, according as the perpendicular falls within or without it's base.*

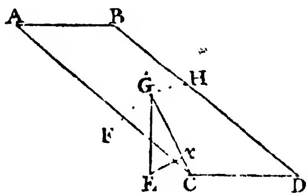
Let *ABDC* represent the body, *G* it's center of gravity; draw *GE* perpendicular to the horizon; join *CG*, and with the radius *CG* describe the circular arc *HGF*; then the body cannot fall over at *C*, unless the



center of gravity describes the circular arc *GF*. Suppose the whole force of gravity applied at *G* (Art. 178.), and take *GE* to represent it; draw *Ex*

perpendicular to CG; then the force GE is equivalent to the two Gx, xE, of which Gx cannot move the body either in the direction GF or GH; and when E falls within the base, xE acts at G in the direction GH; therefore the center of gravity cannot describe the arc GF, that is, the body cannot fall over at C. In the same manner it may be shewn that it cannot fall over at D.

When the perpendicular GE falls without the base, xE acts in the direction GF, and since there is no

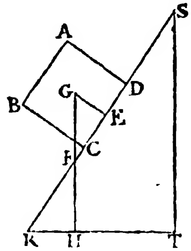


force to counteract this, the center of gravity will move in that direction, or the body will fall.

(186.) Cor. 1. In the same manner it may be shewn, that if a body be placed upon an inclined plane, and the lateral motion be prevented by friction, the body will be sustained or not, according as the perpendicular to the horizon, drawn through it's center of gravity, falls within or without the base.

Ex. Let $ABCD$ represent a cube of uniform density, placed upon the inclined plane RS; G it's center of gravity; draw GE perpendicular to CD, and GFH perpendicular to the horizon; then this body will not be sustained upon the inclined plane, if the angle of the plane's inclination SRT, exceed half a right angle. For

if the $\angle$ *FRH* be greater than half a right angle, the



$\angle$ *RFH*, or *GFE*, is less than half a right angle, and the $\angle$ *FGE* is greater than half a right angle; therefore, *EF* is greater than *EG*, or *EC*, and the body will roll.

(187.) Cor. 2. The higher the center of gravity of a body is, *cœteris paribus*, the more easily it is overturned.

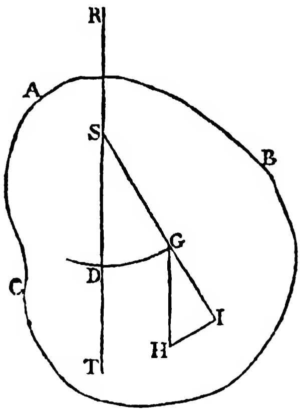
The same construction being made as in the Proposition, the whole weight of the body : that part of the weight which keeps it steady upon it's base, or opposes any power employed to overturn it :: $GE : xE :: GC : CE$; and when *CE* and the whole weight of the body are given, the force which keeps the body steady $\propto \frac{1}{GC}$ (*Alg.* Art. 206.); therefore as *GC* increases, that is, as *GE* increases, the force which keeps the body steady decreases, or the more easily will the body be overturned.

(188.) Cor. 3. When *CE* vanishes with respect to *GC*, the force which keeps the body steady vanishes, and the body may be overturned by a very small force. Thus it is extremely difficult to balance a body upon a *point* placed *under* the center of gravity.

Prop. XLI.

(189.) *If a body be suspended by any point, it will not remain at rest till the center of gravity is in the line which is drawn through that point, perpendicular to the horizon.*

Let S be the point of suspension of the body ABC;



G it's center of gravity; join SG and produce it; through S, and G, draw RST, and GH, perpendiculars to the horizon; then the effort of gravity, to put the body in motion, is the same that it would be, were all the particles collected at G; take GH to represent the force in that direction, and draw HI perpendicular to GI; then the force GH is equivalent to the two GI, IH, of which GI is sustained by the reaction at the point of suspension S, and IH is employed in moving the center of gravity in a direction

perpendicular to SG; therefore the center of gravity cannot remain at rest till IH vanishes; that is, till the angle IGH, or GST, or GSR, vanishes, or SG coincides with RT.

(190.) COR. Hence it follows, that if a body be suspended successively by different points, and perpendiculars to the horizon be drawn through the point of suspension, and passing through the body, the center of gravity will lie in each of these perpendiculars, and consequently, in the point of their intersection.

SECTION VI.

ON THE COLLISION OF BODIES.

(191.) Def. *Hardness*, which is found in different bodies in different degrees, consists in a firm cohesion of the component particles; and that body is said to be harder than another, whose particles require a greater force to separate them. By a *perfectly hard* body we mean one whose parts cannot be separated, or moved one amongst another by any finite force.

(192.) Def. The tendency in a body to recover it's former figure, after having been compressed, is called *elasticity*. That body is said to be more elastic than another, which recovers it's figure with the greater force, supposing the compressing force the same. By a *perfectly elastic* body we mean one which recovers it's figure with a force equal to that which was employed in compressing it.

That such a tendency exists in bodies is evident from a variety of experiments. If an ivory ball,

stained with ink, be brought gently into contact with an unstained ball, the spot received by the latter will be very small, since two spheres touch each other only in a single point; but if one of the balls be made to impinge upon the other, the spot will be enlarged; and the greater the force of impact, the greater will be the surface stained; hence it is manifest, that one, or both of the balls, has been compressed, and afterwards recovered it's spherical figure. Almost all bodies with which we are acquainted are elastic in a greater or less degree; but none perfectly so. In steel balls, the force of elasticity is to the compressing force as 5 to 9; in glass, as 15 to 16; though in all cases, the force of elasticity seems to depend, in some measure, upon the diameter of the ball.

(193.) Def. The impact of two bodies is said to be *direct*, when their centers of gravity move in the right line which passes through the point of impact.

In considering the effects of collision, the bodies are usually supposed to be spheres of uniform density; and in their actions upon each other, not to be affected by gravity, or any other force but that of inertia.

Prop. XLII.

(194.) *If the impact of two perfectly hard bodies be direct, after impact they will either remain at rest, or move on, uniformly, together.*

Since there is no force to turn either body out of the line of direction, they will continue in that line

after impact*. Let A and B be the two bodies, moving in the *same* direction, and let A overtake B; then will A continue to accelerate B's motion, and B will continue to retard A's, till their velocities are equal, at which time they will cease to act upon each other; and since there is no force to separate them, they will move on together, and their common velocity, by the First Law of Motion, will be uniform. When they move in *opposite* directions, if their forces be equal, they will rest after impact; if A's force be greater than B's, the whole velocity of B will be destroyed, and A's not being destroyed, A will communicate velocity to B, and B by it's reaction will retard A, till they move on together, as in the former case.

Prop. XLIII.

(195.) *If the impact of two perfectly hard bodies be direct, their common velocity may be found by dividing the whole momentum before impact, estimated in the direction of either motion, by the sum of the quantities of matter.*

Let A and B be the quantities of matter contained in the bodies, a and b their velocities; then, when they move in the same direction, $Aa+Bb$ is the whole momentum in that direction, before impact. When they move in opposite directions, $Aa-Bb$ is the whole momentum estimated in the direction in which A moves.

* The momenta of the particles in each body are proportional to their weights, since their velocities are equal; these momenta, therefore, will not turn the body to either side of the line passing through the center of gravity (Art. 163.)

In the former case, as much as Aa, the momentum of A, is diminished, so much is Bb, the momentum of B, increased by the impact (Art. 32.); therefore $Aa + Bb$ is equal to the whole momentum after impact.

In the latter case, if Aa be greater than Bb, before the bodies can begin to move together, Bb, the momentum of B, must be destroyed; and therefore A's momentum must be diminished by the quantity Bb (Art. 32.) Thus, when the bodies begin to move in the same direction, $Aa - Bb$ is their whole momentum; and as much momentum as is afterwards communicated to B, so much is lost by A; therefore $Aa - Bb$ is equal to the whole momentum after impact.

If Aa be less than Bb, the momentum after impact, in the direction of B's motion, will be $Bb - Aa$; or, in the direction of A's motion, $Aa - Bb$.

Let v be the common velocity after impact; then $\overline{A+B} \times v$ is the whole momentum; consequently, $\overline{A+B} \times v = Aa \pm Bb$, and $v = \dfrac{Aa \pm Bb}{A+B}$. In which expression, the positive sign is to be used when the bodies move in the same direction before impact, and the negative sign, when they move in *opposite* directions.

(196.) Cor. 1. When the bodies move in opposite directions with equal momenta, they will remain at rest after impact. In this case $Aa - Bb = 0$; therefore $v = 0$.

(197.) Cor. 2. If Bb be greater than Aa, v is negative. This shews that the bodies will move in the directions of B's motion, which was supposed, in the proposition, to be negative.

Prop. XLIV.

(198.) *In the direct impact of two perfectly hard bodies* A *and* B, *estimating the effects in the direction of* A's *motion*, A + B : A :: *the relative velocity of the two bodies* : *the velocity gained by* B. *And* A + B : B :: *their relative velocity* : *the velocity lost by* A.

The same notation being retained; when the bodies move in the same direction, $a-b$ is their relative velocity (Art. 12.); and v, their common velocity after impact, is $\frac{Aa+Bb}{A+B}$ (Art. 195.); therefore, the velocity *gained* by B, or $v-b$, is $\frac{Aa+Bb}{A+B}-b$, or $\frac{Aa-Ab}{A+B}$; hence, $A+B : A :: a-b$: the velocity gained by B. Also, $a-\frac{Aa+Bb}{A+B}$, or $\frac{Ba-Bb}{A+B}$ is the velocity *lost* by A; therefore $A+B : B :: a-b$: the velocity lost by A.

When the bodies move in opposite directions, $a+b$ is their relative velocity (Art. 12.); and $v=\frac{Aa-Bb}{A+B}$ (Art. 195.); also, the velocity communicated to B upon the whole, in the direction of A's motion, is $v+b$, or $\frac{Aa-Bb}{A+B}+b$, that is, $\frac{Aa+Ab}{A+B}$; therefore, $A+B : A :: a+b$: the velocity gained by B.

The velocity lost by A is $a-\frac{Aa-Bb}{A+B}$, or $\frac{Ba+Bb}{A+B}$; therefore, $A+B : B :: a+b$: the velocity lost by A.

Ex. Let the weights of A and B be 10 and 6 * ; their velocities 12 and 8, respectively; then, when they move in the same direction, $10+6 : 10 :: 12-8 : \frac{40}{16}=2\frac{1}{2}$, the velocity gained by B; and $10+6 : 6 :: 12-8 : \frac{24}{16}=1\frac{1}{2}$, the velocity lost by A.

When they move in opposite directions, $12+8$ is their relative velocity; and $10+6 : 10 :: 12+8 : \frac{200}{16}=12\frac{1}{2}$, the velocity gained by B in the direction of A's motion. Also, since it had a velocity 8 in the opposite direction before impact, it's velocity after impact is $4\frac{1}{2}$ in the direction of A's motion. Again, $10+6 : 6 :: 12+8 : \frac{120}{16}=7\frac{1}{2}$, the velocity lost by A.

(199.) Cor. 1. Whilst the relative velocity remains the same, the velocity gained by B, and the velocity lost by A, are unaltered.

(200.) Cor. 2. Hence it also follows that the velocities, gained by B, and lost by A, are the same, whether both bodies are in motion, or A impinges upon B at rest, with a velocity equal to their relative velocity in the former case.

(201.) Cor. 3. If the relative velocity be the same, the momentum communicated is the same, whether A impinges upon B, or B upon A.

Call r the relative velocity; then when A impinges upon B, $A+B : A :: r : \frac{Ar}{A+B}$, the velocity gained

* See Art. 26.

by B; therefore $\frac{ABr}{A+B}$ is the momentum gained by B. When B impinges upon A, $A+B : B :: r : \frac{Br}{A+B}$, the velocity gained by A; therefore $\frac{ABr}{A+B}$ is the momentum gained by A; which is also the momentum gained by B on the former supposition.

Prop. XLV.

(202.) *When the bodies are perfectly elastic, the velocity gained by the body struck, and the velocity lost by the striking body, will be twice as great as if the bodies were perfectly hard.*

Let A and B be the bodies; then, as in Art. 194, A will accelerate B's motion, and B will retard A's, till their velocities are equal; and if they were perfectly hard they would then cease to act upon each other, and move on together; thus, during the first part of the collision, the same effect is produced, that is, the same velocity is gained and lost, as if the bodies were perfectly hard. But, during this period, the bodies are compressed by the stroke, and since they are, by the supposition, perfectly elastic, the force with which each will recover it's former shape is equal to that with which it was compressed; therefore, each body will receive another impulse from the elasticity equal to the former, or B will gain, and A lose, upon the whole, twice as great a velocity as if both bodies had been perfectly hard.

(203.) The same demonstration may be applied to the case where one body is perfectly hard, and the other perfectly elastic.

Prop. XLVI.

(204.) *In the direct impact of two perfectly elastic bodies* A *and* B, A+B : 2A :: *their relative velocity before impact : the velocity gained by* B *in the direction of* A's *motion; and* A+B : 2B :: *their relative velocity : the velocity lost by* A, *in that direction.*

Call r the relative velocity of the bodies; x the velocity gained by B, and y the velocity lost by A, when both bodies are perfectly hard; then $2x$ is the velocity gained by B, and $2y$ the velocity lost by A, when they are perfectly elastic; and

$A+B : A :: r : x$ (Art. 198.); therefore,
$A+B : 2A :: r : 2x$ (*Alg.* Art. 185.), the velocity gained by B.

Again, $A+B : B :: r : y$ (Art. 198.); therefore,
$A+B : 2B :: r : 2y$, the velocity lost by A.

Ex. Let the weights of the bodies be 5 and 4, their velocities 7 and 5; then, when they move in the same direction, $5+4 : 10 :: 7-5 : \frac{20}{9} = 2\frac{2}{9}$, the velocity gained by B; therefore $5+2\frac{2}{9}$, or $7\frac{2}{9}$ is B's velocity after impact. Also, $5+4 : 8 :: 7-5 : \frac{16}{9} = 1\frac{7}{9}$, the velocity lost by A; therefore $7-1\frac{7}{9}$, or $5\frac{2}{9}$, is A's velocity after impact. When they move in opposite

directions, $5+4 : 10 :: 7+5 : \frac{120}{9} = 13\frac{1}{3}$, the velocity gained by B. Also, since it had a velocity 5 in the opposite direction, it's velocity after impact, in the direction of A's motion, is $13\frac{1}{3} - 5$, or $8\frac{1}{3}$. Again, $5+4 : 8 :: 7+5 : \frac{96}{9} = 10\frac{2}{3}$ A's velocity lost; and since it had a velocity 7 before impact, after impact it will move in the opposite direction with a velocity $3\frac{2}{3}$.

(205.) Cor. 1. When $A=B$, the bodies interchange velocities. For, in this case, $A+B=2A=2B$; therefore, the velocity gained by B, and the velocity lost by A, are respectively equal to their relative velocity before impact. Let a and b be their velocities before impact; then, when they move in the same direction, $a-b$ is the velocity gained by B, or lost by A; therefore $a-b+b$, or a, is B's velocity after impact; and $a-\overline{a-b}$, or b, is A's velocity. If b be negative, or the bodies move in opposite directions, $a+b-b$, or a, is B's velocity, and $a-\overline{a+b}$, or $-b$, is A's velocity after impact.

(206.) Cor. 2. If the bodies move in opposite directions with equal quantities of motion, the whole momentum of each will be destroyed during the compression, and an equal one generated by elasticity in the opposite direction; each body will therefore be reflected with a velocity equal to that which it had before impact.

(207.) Cor. 3. The relative velocity of the bodies after impact is equal to their relative velocity before impact.

Let a and b be the velocities of the bodies before impact; p and q their velocities after; then $a-b=q-p$. For, $A+B:2A::a-b:\frac{2A\times\overline{a-b}}{A+B}$, the velocity gained by B; therefore $q=b+\frac{2A\times\overline{a-b}}{A+B}$. Also, $A+B:2B::$ $a-b:\frac{2B\times\overline{a-b}}{A+B}$, the velocity lost by A; therefore $p=$ $a-\frac{2B\times\overline{a-b}}{A+B}$, and $q-p=b-a+\frac{\overline{2A+2B}\times\overline{a-b}}{A+B}$ $=b-a+2a-2b=a-b$. When the bodies A and B move in opposite directions, the sign of b is negative; in other respects the demonstration is the same.

(208.) COR. 4. The sum of the products of each body, multiplied by the square of it's velocity, is the same before and after impact.

The notation in the last Article being retained; $Aa+Bb=Ap+Bq$ (Art. 34.); by transposition, $Aa-Ap$ $=Bq-Bb$; or $A\times\overline{a-p}=B\times\overline{q-b}$. Also $a-b=$ $q-p$ (Art. 207.); or $a+p=q+b$; therefore $A\times\overline{a-p}$ $\times\overline{a+p}=B\times\overline{q-b}\times\overline{q+b}$; or $Aa^2-Ap^2=Bq^2-Bb^2$; therefore $Aa^2+Bb^2=Ap^2+Bq^2$. If any of the quantities, b, p, q, be negative, it's square will be positive, and therefore the conclusion will not be altered.

(209.) COR. 5. If there be a row of equal elastic bodies, A, B, C, D, &c. at rest, and a motion be communicated to A, and thence to B, C, D, &c. they will all remain at rest after the impact, except the last, which will move off with a velocity equal to that with which the first moved.

For A and B will interchange velocities (Art. 205); that is, A will remain at rest, and B move on with A's

velocity. In the same manner it may be shewn that all the others will remain at rest after impact, except the last, which will move off with the velocity communicated to A.

(210.) Cor. 6. If the bodies decrease in magnitude, they will all move in the direction of the first motion, and the velocity communicated to each succeeding body will be greater than that which was communicated to the preceding.

For, $A+B : 2B ::$ A's velocity before impact : the velocity lost by A; and since $2B$ is less than $A+B$, A does not lose it's whole velocity; therefore it will move on after impact in the direction of the first motion. Also, $A+B : 2A ::$ A's velocity before impact : the velocity gained by B; and since $2A$ is greater than $A+B$, the velocity gained by B is greater than A's velocity before impact. In the same manner it may be shewn that B, C, D, &c. will move on in the direction of the first motion; and that the velocity communicated to each will be greater than that which was communicated to the preceding body.

(211.) Cor. 7. If the bodies increase in magnitude, they will all be reflected back, except the last, and the velocity communicated to each succeeding body will be less than that which was communicated to the preceding.

For, in this case, $2B$ is greater than $A+B$; therefore, A loses more than it's whole velocity, or it will move in the contrary direction. Also, $2A$ is less than $A+B$; therefore, the velocity gained by B is less than A's velocity before impact. In the same manner it may be shewn that B, C, D, &c. will be reflected;

and that the velocity communicated to each will be less than that which was communicated to the preceding body.

(212.) Cor. 8. The velocity thus communicated from A through B to C, when B is greater than one of the two A, C, and less than the other, exceeds the velocity which would be communicated immediately from A to C.

Let a represent A's velocity; then

$A+B : 2A :: a : \frac{2Aa}{A+B}$, the velocity of B; and

$B+C : 2B :: \frac{2Aa}{A+B} : \frac{2Aa}{A+B} \times \frac{2B}{B+C}$, the velocity communicated from B to C.

Again, $A+C : 2A :: a : \frac{2Aa}{A+C}$, the velocity communicated immediately from A to C. Hence it follows, that the velocity communicated to C, by means of B, is greater than that which would be communicated to it immediately, if $\frac{2Aa}{A+B} \times \frac{2B}{B+C}$ be greater than $\frac{2Aa}{A+C}$; that is, if $A+C$ be greater than $\frac{\overline{A+B} \times \overline{B+C}}{2B}$, or $2A+2C$ greater than $A+C+B+\frac{AC}{B}$; or $A+C$ greater than $B+\frac{AC}{B}$. Suppose $A=B+x$, $C=B+y$; then $A+C=2B+x+y$, and $B+\frac{AC}{B} = B + \frac{B^2+Bx+By+xy}{B} = 2B+x+y+\frac{xy}{B}$; therefore, the velocity communicated to C by means

of B, is greater than the velocity communicated to it without B, if $2B+x+y$ be greater than $2B+x+y+\frac{xy}{B}$, which will always be the case when xy is negative, or when x and y have different signs; that is, when B is less than one of the bodies, A, C, and greater than the other*.

(213.) Cor. 9. If the bodies be in geometrical progression, the velocities communicated to them will be in geometrical progression; and when there are n such bodies, whose common ratio is r, the velocity of the first : the velocity of the last :: $\overline{1+r}|^{n-1} : 2^{n-1}$.

Let A, Ar, Ar^2, Ar^3, &c. be the bodies; a, b, c, d, &c. the velocities successively communicated to them; then

$A + Ar : 2A :: a : b$, or

$1 + r : 2 :: a : b$; and in the same manner,

$1 + r : 2 :: b : c$

$1 + r : 2 :: c : d$, &c.

therefore $a : b :: b : c :: c : d$ &c. Also, by composition, $\overline{1+r}|^{n-1} : 2^{n-1} :: a$: the velocity of the last.

(214.) Cor. 10. If the number of mean proportionals, interposed between two given bodies A and X, be increased without limit, the ratio of A's velocity to the velocity thus communicated to X will approximate to the ratio of $\sqrt{X} : \sqrt{A}$ as it's limit.

Let A, B, C, D, ... X be the bodies; a, b, c, d, ... x the velocities communicated to them. Then since the number of bodies interposed between

* The velocity communicated from A through B to C, is a maximum when A, B, and C, are in geometrical progression. (*Flux.* Art. 21. Ex. 11.)

A and X is increased without limit, their differences will be diminished without limit; let $A+z=B$; then

$$2A+z : 2A :: a : b$$

$$\text{or } A+\frac{z}{2} : A :: a : b$$

$$\text{* and } A+\frac{z}{2} : A :: \sqrt{A+z} : \sqrt{A} :: \sqrt{B} : \sqrt{A};$$

$$\text{therefore, } \sqrt{B} : \sqrt{A} :: a : b$$

$$\text{in the same manner, } \sqrt{C} : \sqrt{B} :: b : c$$

$$\sqrt{D} : \sqrt{C} :: c : d$$

&c.

$$\text{comp. } \sqrt{X} : \sqrt{A} :: a : x.$$

Cor. The conclusion is the same when the intermediate bodies vary according to any other law, if the difference of the succeeding bodies, in every part of the series, be evanescent.

Prop. XLVII.

(215.) *In the direct impact of two perfectly elastic bodies* A *and* B, *if the compressing force be to the force of elasticity* :: 1 : m, *then* $A+B : \overline{1+m} \times A$ :: *their relative velocity before impact* : *the velocity gained by* B *in the direction of* A's *motion. And* $A+B : \overline{1+m} \times B$ :: *their relative velocity before impact* : *the velocity lost by* A, *in that direction.*

By reasoning, as in Art. 202, it appears that the

* Since $\overline{A+\frac{z}{2}}\big|^2 : A^2 :: A^2+Az+\frac{z^2}{4} : A^2 :: A+z+\frac{z^2}{4A} : A :: B+\frac{z^2}{4A} : A$, the ratio of $\overline{A+\frac{z}{2}}\big|^2 : A^2$, when z is continually diminished, approximates to the ratio of $B : A$, and consequently, the ratio of $A+\frac{z}{2} : A$ approximates to the ratio of $\sqrt{B} : \sqrt{A}$ as it's limit.

velocity gained by B, and the velocity lost by A during the compression, are the same as if the bodies were perfectly hard; and the velocity communicated by the elasticity is to the velocity communicated by the compression :: m : 1. Call r the relative velocity before impact, x the velocity gained by B, and y the velocity lost by A, during the compression; then $\overline{1+m} \times x$ is the velocity gained by B, and $\overline{1+m} \times y$ the velocity lost by A, upon the whole. Now

$$A+B : A :: r : x \text{ (Art. 198.)},$$

$$\text{and } A+B : B :: r : y;$$

therefore, $A+B : \overline{1+m} \times A :: r : \overline{1+m} \times x$, the velocity gained by B;

and $A+B : \overline{1+m} \times B :: r : \overline{1+m} \times y$, the velocity lost by A.

(216.) Cor. 1. The relative velocity before impact : the relative velocity after impact :: 1 : m.

Let a and b be the velocities of the two bodies before impact, p and q their velocities after; then

$A+B : \overline{1+m} \times A :: a-b : \dfrac{\overline{1+m} \times A \times \overline{a-b}}{A+B}$, the velocity gained by B;

$$\text{therefore, } q = b + \frac{\overline{1+m} \times A \times \overline{a-b}}{A+B};$$

$$\text{in the same manner, } p = a - \frac{\overline{1+m} \times B \times \overline{a-b}}{A+B};$$

$$\text{hence, } q-p = b-a+\frac{\overline{1+m} \times \overline{A+B} \times \overline{a-b}}{A+B}, \text{ or}$$

$b-a+a-b+m \times \overline{a-b}$, i. e. $m \times \overline{a-b}$, is the relative velocity after impact; and $a-b : m \times \overline{a-b} :: 1 : m$.

When the bodies move in opposite directions, the sign of b is negative.

(217.) Cor. 2. Hence it appears that if the velocities of the bodies before and after impact be known, the elastic force is known.

(218.) Cor. 3. If A impinge upon B at rest, A will remain at rest after impact when $A : B :: m : 1$.

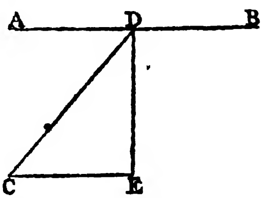
In this case A loses it's whole velocity, and $A+B : \overline{1+m} \times B :: a$: the velocity lost by A; therefore $A+B = \overline{1+m} \times B$, and $A = mB$; consequently, $A : B :: m : 1$.

(219.) Cor. 4. The momentum communicated is the same, whether A impinges upon B, or B upon A, if the relative velocity be the same. This is the case when the bodies are perfectly hard (Art. 201.); and the effect produced in elastic bodies is in a given ratio to that which is produced when the bodies are perfectly hard.

Prop. XLVIII.

(220.) *When a perfectly hard body impinges obliquely on a perfectly hard and immoveable plane* AB, *in the direction* CD, *after impact it will move along the plane, and the velocity before impact : the velocity after :: radius : the cosine of the angle* CDA.

Take CD to represent the motion of the body before



impact; draw CE parallel, and DE perpendicular to AB.

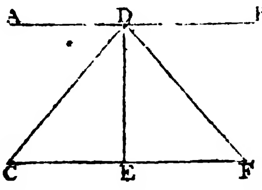
Then CD may be resolved into the two CE, ED, (Art. 43.), of which ED is wholly employed in carrying the body in a direction perpendicular to the plane; and since the plane is immoveable, this motion will be wholly destroyed, (See Art. 116.) The other motion CE, which is employed in carrying the body parallel to the plane, will not be affected by the impact; and consequently, there being no force to separate the body and the plane, the body will move along the plane; and it will describe $DB = CE$ in the same time that it described CD before impact; also, these spaces are uniformly described (Art. 27.); consequently, the velocity before impact : the velocity after :: CD : CE :: radius : sin. $\angle CDE$ :: radius : cos. $\angle CDA$.

(221.) Cor. The velocity before impact : the difference between the velocity before and the velocity after, that is, the velocity lost :: radius : rad. - cos. $\angle CDA$ :: rad. : the versed sine of the angle CDA.

Prop. XLIX.

(222.) *If a perfectly elastic body impinge upon a perfectly hard and immoveable plane* AB, *in the direction* CD, *it will be reflected from it in the direction* DF, *which makes, with* DB, *the angle* BDF *equal to the angle* ADC.

Let CD represent the motion of the impinging



body; draw CF parallel, and DE perpendicular to

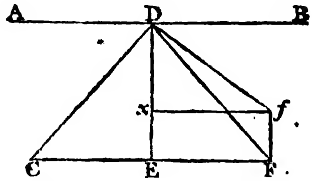
AB; make $EF=CE$, and join DF. Then the whole motion may be resolved into the two CE, ED, of which CE is employed in carrying the body parallel to the plane, and must therefore remain after the impact; and ED carries the body in the direction ED, perpendicular to the plane; and since the plane is immoveable, this motion will be destroyed during the compression, and an equal motion will be generated in the opposite direction by the force of elasticity. Hence it appears, that the body at the point D has two motions, one of which would carry it uniformly from D to E, and the other from E to F, in the same time, viz. in the time in which it described CD before the impact; it will, therefore, describe DF in that time (Art. 38.) Also, in the triangles CDE, EDF, CE is equal to EF, the side ED is common, and the $\angle CED$ is equal to the $\angle DEF$; therefore, the $\angle CDE =$ the $\angle EDF$; hence, the $\angle CDA =$ the $\angle FDB$.

(223.) Cor. 1. Since $CD=DF$, and these are spaces uniformly described in equal times, before and after the impact, the velocity of the body after reflection is equal to it's velocity before incidence.

(224.) Cor. 2. If the body and plane be imperfectly elastic, take $DE : Dx ::$ the force of compression : the force of elasticity; draw xf parallel and equal to EF, join Ff, Df; then the two motions which the body has at D are represented by Dx, xf*, and the body will describe Df, after reflection, in the same time that it described CD before incidence; therefore, the

* Here we suppose the common surface of the body and plane, during the impact, to remain parallel to AB, in which case there is no cause to accelerate or retard the motion CE (See Art. 116.)

velocity before incidence : the velocity after reflection

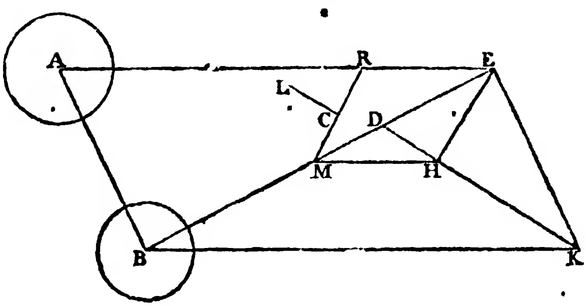


:: $CD : Df$ :: $DF : Df$ :: sin. DfF, or sin. of it's supplement EDf : sin. DFf, or sin. FDE :: sin. EDf : sin. EDC.

Prop. L.

(225.) *Having given the radii of two spherical bodies moving in the same plane, their velocities, and the directions in which they move, to find the plane which touches them both at the point of impact.*

Let AE, BE, meeting in E, be the directions in which the bodies A and B move; and let AE and BD be spaces uniformly described by them in the same time; complete the parallelogram $ABKE$; join KD, and with the center E and radius equal to the sum of the radii of the two bodies, describe a circular



arc cutting KD in H; join EH, and complete the parallelogram $EHMR$. Then R and M will be the

places of the centers of the two spheres when they meet; and if RC be taken equal to the radius of the sphere A, the plane CL, which is drawn through C perpendicular to MR, will be the plane required.

Since MH is parallel to AE or BK, the triangles DMH, DBK, are similar, and $BK : BD :: MH : MD$; or $AE : BD :: RE : MD$; therefore $AE : BD :: AR : BM$ (Euc. 19. v.); and since AE and BD are spaces described in the same time by the uniform motions of A and B, AR and BM, which are proportional to them, will be described in the same time; when, therefore, the center of the body A is in R, the center of the body B is in M, and the distance $MR = HE =$ the sum of the radii of the bodies; hence, they will be in contact when they arrive at those points. Also, MR which joins their centers will pass through the point of contact; and LC will be a tangent to them both.

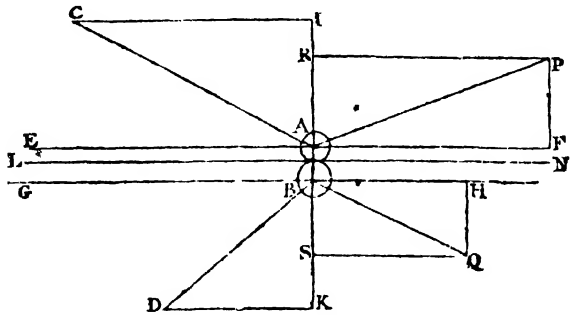
Prop. LI.

(226.) *Having given the motions, the quantities of matter, and the radii of two spherical bodies which impinge obliquely upon each other, to find their motions after impact.*

Let LN be the plane which touches the bodies at the point of impact; produce AB, which joins the centers of the bodies, indefinitely both ways; through the centers A and B, draw EAF, GBH, parallel to LN; let CA, DB, represent the velocities of the bodies before impact; resolve CA into the two CI, IA*, of which CI is parallel, and IA perpendicular to LN; also resolve

* See Art. 43.

DB into two, *DK* parallel to *LN*, and *KB* perpendicular to it. Then *CA* and the angle *CAI*, which the direction of *A*'s motion makes with *AI* perpendicular to *LN*, being known, *CI* and *IA* are known; in the same manner, *DK* and *KB* are known. Now *CI*, *DK*, which are parallel to the plane *LN*, will not



be affected by the impact; and *IA*, *KB*, which are perpendicular to it, are the velocities with which the bodies impinge directly upon each other, and their effects may be calculated by Prop. 44, when the bodies are perfectly hard; and by Prop. 47, when they are elastic. Let *AR* and *BS* be the velocities of the bodies after impact, thus determined; take *AF*=*CI*, and *BH*=*DK*; complete the parallelograms *RF*, *SH*, and draw the diagonals *AP*, *BQ*; then the bodies will describe the lines *AP*, *BQ*, after impact, and in the same time that they describe *CA*, *DB*, before impact.

SECTION VII.

ON THE RECTILINEAR MOTIONS OF BODIES ACCELERATED OR RETARDED BY UNIFORM FORCES.

Prop. LII.

(227.) *If a body be impelled in a right line by an uniform force, the velocity communicated to it is proportional to the time of it's motion**.

The accelerating force is measured by the velocity uniformly generated in a given time (Art. 21.), and in this case, the force is invariable, by the supposition; therefore, equal increments of velocity are always generated in equal times (Art. 20.); and since a body, by the first law of motion, retains the increments of velocity thus communicated to it, if, in the time t, the velocity a be generated, in the time mt the velocity ma is generated; that is, the velocity generated is proportional to the time (*Alg.* Art. 193.).

* By *force*, in this and the following Propositions, we understand the *accelerating force*, no regard being paid to the quantity of matter moved, unless it be expressly mentioned. Also, the direction in which the force acts, to generate or destroy velocity, is supposed to coincide with the direction of the motion.

Prop. LIII.

(228.) *If bodies be impelled in right lines by different uniform forces, the velocities generated in any times are proportional to the forces and times jointly.*

Let F and f be the forces, T and t the times of their action, V and v the velocities generated; also, let x be the velocity generated by the force f in the time T; then,

$$V : x :: F : f \quad \text{(Art. 21.)};$$
$$x : v :: T : t \quad \text{(Art. 227.)};$$

comp. $V : v :: FT : ft$; that is, the velocities generated are proportional to the forces and times jointly (*Alg.* Art. 195.).

Ex. If a force, represented by unity, generate a velocity represented by $2m$, in one second of time, what velocity will the force F generate in T seconds?

Since $V \propto FT$, we have $1 \times 1 : FT :: 2m : 2mFT$, the velocity required.

(229.) Cor. Since $V \propto FT$, $T \propto \frac{V}{F}$ (*Alg.* Art. 205.)

Prop. LIV.

(230.) *If a body's motion be retarded by an uniform force, the velocity destroyed in any time is equal to that which would be generated in the same time, were the motion accelerated by the same force.*

The force impressed is the same, by the supposition, whether the body move in the direction of the force, or in the opposite direction; therefore, the velocity generated in the former case, is equal to the velocity destroyed, in the same time, in the latter (Art. 29.)

(231.) Cor. 1. Hence, the velocity destroyed by an uniform force is proportional to the time of it's action.

For, the velocity generated by the action of the force is in that ratio (Art. 227.)

(232.) Cor. 2. The velocities destroyed by different uniform forces, are proportional to the forces and times jointly (Art. 228.)

Prop. LV.

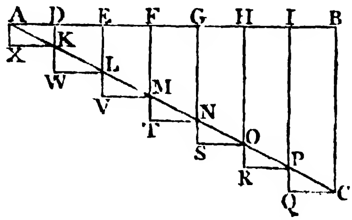
(233.) *If a body be moved through any space, from a state of rest, by the action of an uniform force, and then be projected in the opposite direction with the velocity acquired, and move till that velocity is destroyed, the whole spaces described in the two cases are equal.*

The velocity generated in any time, is equal to the velocity destroyed in the same time by the action of the same force (Art. 230.); hence, the whole times of motion, in the two cases, are equal; also, if equal times be taken, from the beginning of the motion in the former case, and from the end of the motion in the latter, the velocities at those instants are equal. Since then the whole times of motion are equal, and also the velocities at all corresponding points of time, the whole spaces described are equal.

Prop. LVI.

(234.) *If a body be moved from a state of rest by the action of an uniform force, the space described, reckoning from the beginning of the motion, varies as the square of the time, or as the square of the last acquired velocity.*

Take AB to represent the time of the body's motion; draw BC at right angles to AB, and let BC



represent the last acquired velocity; join AC; divide the time AB into small equal portions AD, DE, EF, FG, &c.; and from the points D, E, F, G, &c. draw DK, EL, FM, GN, &c. parallel to BC, meeting AC in the points K, L, M, N, &c.; complete the parallelograms DX, EW, FV, GT, &c.

Then, in the similar triangles ABC, ADK, we have $AB : AD :: BC : DK$; and since BC represents the velocity acquired in the time AB, DK represents the velocity acquired in the time AD (Art. 227.); in the same manner it appears, that EL, FM, GN, &c. represent the velocities generated in the times AE, AF, AG, &c. Now, if the body move with the uniform velocity DK, during the time AD, and with the uniform velocities EL, FM, GN, &c. during the times DE, EF, FG, &c. respectively, the spaces described may properly be represented by the rectangles DX, EW, FV, GT, &c. (Art. 16.); therefore, the whole space described, on this supposition, will be represented by the sum of the rectangles, or by the triangle ABC, together with the sum of the triangles AXK, KWL, LVM, MTN, &c. that is, because the bases of these small triangles are respectively equal

to IB, and the sum of their altitudes is equal to BC, by the triangle ABC, together with half the rectangle BQ. Let the intervals AD, DE, EF, FG, &c. be diminished without limit with respect to AB, and the rectangle BQ is diminished without limit with respect to the triangle ABC; or $ABC+\frac{1}{2}BQ$ approaches to ABC as it's limit; therefore, when the motion of the body is constantly accelerated, the space described is represented by the area of the triangle ABC. The space described in any other time AG, reckoning from the beginning of the motion, is represented, on the same scale, by the area of the triangle AGN; and because these triangles are similar, the space described in the time AB : the space described in the time AG :: $AB^2 : AG^2$.

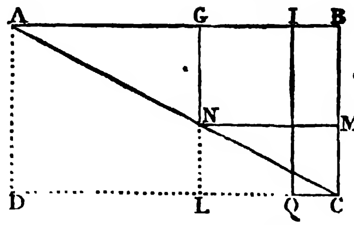
Also, BC, GN, represent the velocities generated in the times AB, AG; and from the same similar triangles, the space described in the time AB : the space described in the time AG :: $BC^2 : GN^2$.

(235.) Ex. If a body be accelerated from a state of rest by an uniform force, and describe m feet in the first second of time, it will describe $4m$, $9m$, $16m$, ... mT^2 feet, in the 2, 3, 4, ... T first seconds.

(236.) Cor. 1. The space described, reckoning from the beginning of the motion, is half that which would be described in the same time with the last acquired velocity continued uniform.

Complete the parallelogram BD; then, it appears from the Proposition, that the space described in the time AB, reckoning from the beginning of the motion : the space described in the time IB with the

uniform velocity *BC* :: the triangle *ABC* : *BQ*. Also,



the space described in the time *IB*, with the uniform velocity *BC* : the space described in the time *AB*, with the same uniform velocity :: *IB* : *AB* (Art. 13.) :: *BQ* : *BD*; and by compounding these two proportions, we have the space described in the time *AB*, when the body's motion is accelerated from a state of rest : the space described in the same time with the last acquired velocity continued uniform :: the triangle *ABC* : the rectangle *BD* :: 1 : 2*.

(237.) Cor. 2. The space described in the time *GB* is represented by the area *GBCN*; or, if *NM* be drawn parallel to *GB*, by the rectangle *GM* together with the triangle *NMC*. Now, *GM* represents the space which a body would describe in the time *GB*, with the uniform velocity *GN*; and the triangle *NMC*, which is similar to the triangle *ABC*, represents the

* This proof has been misunderstood; it amounts to this: The rectangle *BQ* represents the space uniformly described, with the velocity *BC*, in the time *BI*, on the same scale that the triangle *ABC* represents the space through which the body is drawn, by the action of the uniform force, in the time *AB*; and also, on the same scale that *DB* represents the space uniformly described in the time *AB*, with the velocity *BC*; consequently, the spaces described, when the body's motion is accelerated from rest for the time *AB*, and when the velocity *BC* remains uniform for the same time, are represented, *on the same scale*, by the triangle *ABC* and the rectangle *BD*.

space through which the body would be moved from a state of rest, by the action of the force, in the time *GB*; thus, the space described in any time, when a body is projected in the direction of the force, is equal to the space which it would have described, in that time, with the first velocity continued uniform, together with the space through which it would have been moved from a state of rest, in the same time, by the action of the force.

(238.) Cor. 3. If a body be projected in a direction opposite to that in which the uniform force acts, with the velocity *BC*, and move till that velocity is destroyed, the whole time of it's motion is represented by *BA*, (Art. 230.), and the space described by the area *ABC* (Art. 233.)

Also, the space described in the time *BG* is represented, on the same scale, by the area *BGNC*, that is, by the rectangle *BL* diminished by the triangle *CLN*, or *CNM*. Thus it appears, that the space described in the time *BG*, is equal to that which would have been described with the first velocity continued uniform during that time, diminished by the space through which the body would have been moved from a state of rest, in the same time, by the action of the uniform force.

Prop. LVII.

(239.) *When bodies are put in motion by uniform forces, the spaces described in any times, reckoning from the beginning of the motion in each case, are proportional to the times and last acquired velocities jointly.*

Let S and s be the spaces described in the times T and t; V and v the velocities acquired; then $2S$ and $2s$ are the spaces which would be described in the times T and t, with the uniform velocities V and v (Art. 236.); and the spaces described with uniform velocities are proportional to the times and velocities jointly (Art. 14.); hence,

$$2S : 2s :: TV : tv,$$

$$\text{or } S : s :: TV : tv \text{ (*Alg.* Art. 184.)};$$

that is, $S \propto TV$ (*Alg.* Art. 195.)

(240.) Cor. Hence, the times vary as the spaces directly, and the last acquired velocities inversely.

PROP. LVIII.

(241.) *The spaces described, reckoning from the beginning of the motions, vary also as the forces and squares of the times; or as the squares of the velocities directly, and the forces inversely.*

In general, $S \propto TV$ (Art. 239.); and $V \propto FT$ (Art. 228.); hence, $TV \propto FT^2$ (*Alg.* Art. 203.); therefore, $S \propto FT^2$. Also, $T \propto V$ therefore, $TV \propto \frac{V^2}{F}$, and consequently, $S \propto \frac{V}{F}$.

(242.) Cor. 1. If T be given, $S \propto F$; that is, the spaces described in equal times, by bodies which are put in motion by uniform forces, are proportional to those forces.

(243.) Cor. 2. Since $S \propto \frac{V^2}{F}$ we have $V^2 \propto FS$ (*Alg.* Art. 203.); that is, the squares of the velocities communicated are proportional to the forces and spaces described jointly.

(244.) Cor. 3. If V be given, $S \propto \frac{1}{F}$.

(245.) Cor. 4. When bodies in motion are retarded by uniform forces, and move till their whole velocities are destroyed, the spaces described vary as the forces and squares of the times; or, as the squares of the first velocities directly and the forces inversely.

For, the time in which any velocity is destroyed, is equal to the time in which it would be generated by the same force; also, the spaces described, on supposition that the body in the latter case is moved from a state of rest, are equal (Art. 233.); therefore, the same expressions which represent the relations of the forces, spaces, times, and velocities, in accelerated motions, represent them also when the motions are retarded, and the bodies move till their whole velocities are destroyed.

Thus, when equal bodies are made to impinge upon banks of earth, sand, &c. where the retarding forces are invariable, the depths to which they sink, or the whole spaces described, are as the squares of the first velocities directly and the forces inversely; and the resisting forces are as the squares of the first velocities directly and the spaces inversely.

Prop. LIX.

(246.) *If a body be moved from a state of rest by the action of an uniform force, the spaces described in equal successive portions of time, reckoned from the beginning of the motion, are as the odd numbers* 1, 3, 5, 7, 9, &c.

If m be the space described in the first portion of time, $4m$ will be the space described in the two first portions (Art. 235.); therefore, $4m-m$, or $3m$, will be the space described in the second portion alone. Also, $9m$ will be the space described in the three first portions of time, and consequently, $9m-4m$, or $5m$, will be the space described, in the third portion, &c. Thus the spaces described in the equal successive portions of time, are m, $3m$, $5m$, $7m$, $9m$, &c. which are as the odd numbers 1, 3, 5, 7, 9, &c.

(247.) Cor. When a body is retarded by an invariable force, the spaces described in equal portions of time, reckoning from the end of the motion, are as the odd numbers 1, 3, 5, 7, 9, &c.

For, when a body moves till it's whole motion is destroyed by an uniform force, the space described in any time is equal to that which would be described in the corresponding time, were the body moved from a state of rest by the action of the same force (See Art. 233.)

Prop. LX.

(248.) *The force of gravity, at any given place, is an uniform force, which always acts in a direction perpendicular to the horizon, and accelerates all bodies equally.*

The same body will, by it's gravity, always produce the same effect under the same circumstances; thus, it will, at the same place, bend the same spring in the same degree; it will also fall through the same space in the same time, if the resistance of the air be removed;

therefore, the force of gravity is uniform. Also, all bodies which fall freely by this force, descend in lines perpendicular to the horizon; and, in an exhausted receiver, they *all* fall through the same space in the same time; consequently, gravity acts in a direction perpendicular to the horizon (Art. 29.), and accelerates *all* bodies equally (Art. 242.)

It is found by experiments made on the descent of heavy bodies, and on the oscillations of bodies in small circular arcs (Art. 302.), that every body which falls freely *in vacuo* by the force of gravity, descends from rest through $16\frac{1}{12}$ feet in one second of time.

This fact being established, every thing relating to the descent of bodies when they are accelerated by the force of gravity, and to their ascent when they are retarded by that force, supposing the motions to be *in vacuo*, may be deduced from the foregoing Propositions.

1st. When a body falls by the force of gravity, the velocity acquired in any time, as T'', is such as would carry it uniformly over $2mT$ feet in $1''$; where $m = 16\frac{1}{12}$.

Since a body falls $16\frac{1}{12}$ feet in $1''$, it acquires a velocity which would carry it uniformly through $32\frac{1}{6}$ feet in $1''$ (Art. 236.); and when a body is accelerated by a given invariable force, the velocity generated is proportional to the time (Art. 226.); therefore, $1'' : T'' :: 32\frac{1}{6} : 32\frac{1}{6}\,T$, the velocity acquired in T''; that is, the velocity acquired is such as would carry the body uniformly over $32\frac{1}{6}\,T$ feet in $1''$. Let V be the velocity acquired, and $m = 16\frac{1}{12}$, then $V = 2mT$.

2d. The space fallen through in T'', reckoned from the beginning of the motion, is mT^2 feet.

For $S \propto T^2$ (Art. 234.); therefore, $1^2 : T^2 :: m : mT^2$, the space described in T''. That is, $S = mT^2$.

Ex. 1. In $3''$ a body falls $9m$, or $9 \times 16\frac{1}{12} = 144\frac{3}{4}$ feet.

Ex. 2. In $\frac{1}{2}''$ a body falls $\frac{m}{4}$, or $16\frac{1}{12} \times \frac{1}{4} = 4\frac{1}{48}$ feet.

3d. The space fallen through to acquire the velocity V, is $\frac{V^2}{4m}$ feet.

For, $S \propto V^2$ (Art. 234.); therefore, $\overline{2m}|^2 : V^2 :: m : S$, and $S = \frac{V^2}{4m}$ feet.

Ex. If a body fall from rest till it has acquired a velocity of 20 feet per second, the space fallen through is $\frac{20 \times 20}{64\frac{1}{3}}$, or 6.21 feet, nearly.

From the three preceding expressions, $V = 2mT$; $S = mT^2$; and $S = \frac{V^2}{4m}$; any one of the quantities S, T, V, being given, the other two may be found.

Ex. To find the time in which a body will fall 90 feet; and the velocity acquired.

Since $S = mT^2$, $T^2 = \frac{S}{m}$, and $T = \sqrt{\frac{S}{m}}$; in this case $T = \sqrt{\frac{90}{16\frac{1}{12}}} = 2.36$ seconds, nearly.

Also, $S = \frac{V^2}{4m}$; therefore, $V = 2\sqrt{mS}$; in this case, $V = 2\sqrt{16\frac{1}{12} \times 90} = 76$ feet per second, nearly.

4th. If a body fall from rest by the force of gravity, the spaces described in any equal successive portions of time, reckoning from the beginning of the motion, are as the numbers 1, 3, 5, 7, &c. Thus, the spaces fallen through in the 1st, 2d, 3d, 4th seconds, are $16\frac{1}{12}$, $3 \times 16\frac{1}{12}$, $5 \times 16\frac{1}{12}$, $7 \times 16\frac{1}{12}$ feet,* respectively. Also, if a body, projected upwards, move till it's whole velocity is destroyed, the spaces described in equal successive portions of time are as the numbers 1, 3, 5, 7, &c. taken in an inverted order. Thus, if the velocity be wholly destroyed in 4″, the spaces described in the 1st, 2d, 3d, 4th seconds, are $7 \times 16\frac{1}{12}$, $5 \times 16\frac{1}{12}$, $3 \times 16\frac{1}{12}$, $16\frac{1}{12}$ feet, respectively.

5th. If a body begin to move in the direction of gravity with any velocity, the whole space described in any time is equal to the space through which the first velocity would carry the body, together with the space through which it would fall by the force of gravity in that time (Art. 237.).

Ex. If a body be projected perpendicularly downwards, with a velocity of 20 feet per second, to find the space described in 4″.

The space described in 4″, with the first velocity, is 4×20, or 80 feet; and the space fallen through in 4″, by the action of gravity, is $16\frac{1}{12} \times 16$, or $257\frac{1}{3}$ feet; therefore, the whole space described is $80 + 257\frac{1}{3}$, or $337\frac{1}{3}$ feet.

6th. If a body be projected perpendicularly upwards, the height to which it will ascend in any time is equal to the space through which it would move

with the first velocity continued uniform, diminished by the space through which it would fall by the action of gravity in that time (Art. 238.).

Ex. 1. To what height will a body rise in $3''$, if projected perpendicularly upwards with a velocity of 100 feet per second?

The space which the body would describe in $3''$, with the first velocity, is 300 feet; and the space through which the body would fall by the force of gravity in $3''$ is $16\frac{1}{12} \times 9$, or $144\frac{3}{4}$ feet; therefore the height required is $300 - 144\frac{3}{4}$, or $155\frac{1}{4}$ feet.

Ex. 2. If a body be projected perpendicularly upwards with a velocity of 80 feet per second, to find it's place at the end of $6''$.

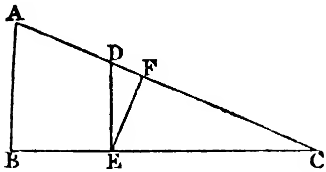
The space which would be described in $6''$, with the first velocity, is 480 feet, and the space fallen through in the same time is $16\frac{1}{12} \times 36$, or 579 feet; therefore the distance of the body from the point of projection, at the end of $6''$, is $480 - 579$, or -99 feet. The negative sign shews that the body will be below the point of projection (See *Alg.* Art. 472.).

Prop. LXI.

(249.) *The force which accelerates or retards a body's motion upon an inclined plane, is to the force of gravity, as the height of the plane is to it's length.*

Let AC be the plane, BC it's base, parallel to the horizon, AB it's perpendicular height, D the place of a body upon it. From the point D draw DE parallel to AB, and take DE to represent the force of gravity; from E draw EF perpendicular to AC.

Then the whole force DE is equivalent to the two



DF, FE, of which FE is perpendicular to the plane, and, consequently, is supported by the plane's reaction (Art. 116.); the other force DF, not being affected by the plane, is wholly employed in accelerating or retarding the motion of the body in the direction of the plane; therefore, the accelerating force : the force of gravity :: $DF : DE$ :: (from the similar triangles DEF, ABC) $AB : AC$.

(250.) Cor. 1. Since the accelerating force, on the same plane, is in a given ratio to the force of gravity, it is an uniform force.

(251.) Cor. 2. If H be the height of an inclined plane, L it's length, and the force of gravity be represented by unity, the accelerating force on the inclined plane is represented by $\frac{H}{L}$.

For, the accelerating force : the force of gravity (1) :: $H : L$; therefore the accelerating force $= \frac{H}{L}$.

(252.) Cor. 3. Since $H : L$ :: the sine of the plane's inclination : the radius, $\frac{H}{L}$, or the accelerating force, varies as the sine of the plane's inclination to the horizon.

(253.) Cor. 4. If a body fall down an inclined

plane, the velocity V, generated in T'', is such as would carry it uniformly over $\frac{H}{L} \times 2mT$ feet in $1''$; where $m = 16\frac{1}{12}$.

In general, $V \propto FT$ (Art. 228.); therefore, the velocity acquired when a body falls by the force of gravity : the velocity acquired on the inclined plane :: the product of the numbers which represent the force and time in the former case : the product of the numbers which represent them in the latter *; also, the force of gravity being represented by unity, the accelerating force upon the plane is $\frac{H}{L}$, and the velocity generated by the force of gravity in $1''$ is $2m$; therefore, $2m : V :: 1 \times 1 : \frac{H}{L} \times T$; and $V = \frac{H}{L} \times 2mT$ †.

Ex. Thus, if the length of an inclined plane be twice as great as it's height, a body which falls down this plane will, in $3''$, acquire a velocity of $\frac{1}{2} \times 32\frac{1}{6} \times 3$, or $48\frac{1}{4}$ feet per second.

(254.) Cor. 5. The space fallen through in T'', from a state of rest, is $\frac{H}{L} \times mT^2$ feet.

In general, $S \propto FT^2$ (Art. 241.); therefore, the space through which a body falls by the action of gravity in $1''$: the space through which it falls down the inclined plane in T'' :: the product of the numbers which represent the force and square of the time in the

* See Note, page 12.

† In this, and the following Articles, the planes are supposed to be perfectly smooth, and the resistance of the air inconsiderable.

former case : the product of the numbers which represent them in the latter; or, if S be the space described upon the plane, $m : S :: 1 \times 1^2 : \frac{H}{L} \times T^2$, and $S = \frac{H}{L} \times mT^2$.

Ex. 1. If $L = 2H$, the space through which a body falls in 3″ is $\frac{1}{2} \times 16\frac{1}{12} \times 9$, or $72\frac{3}{8}$ feet.

Ex. 2. To find the time in which a body will descend 12 feet down this plane.

Since $S = \frac{H}{L} \times mT^2$, $T^2 = \frac{L \times S}{H \times m}$ = (in this case) $\frac{2}{1} \times 12 \times \frac{1}{16\frac{1}{12}} = 1.49$; and $T = 1.2$, nearly.

(255.) Cor. 6. The space through which a body must fall, from a state of rest, to acquire the velocity V, is $\frac{L}{H} \times \frac{V^2}{4m}$ feet.

In general, $S \propto \frac{V^2}{F}$ (Art. 241.); therefore, the space through which the body falls by the force of gravity : the space through which it falls down the plane :: $\frac{V^2}{F}$ in the former case : $\frac{V^2}{F}$ in the latter; and if m $(16\frac{1}{12})$ be the space fallen through by the action of gravity, $2m$ is the velocity acquired; hence, $m : S :: \frac{4m^2}{1} : \frac{L}{H} \times V^2$; and $S = \frac{L}{H} \times \frac{V^2}{4m}$.

Ex. 1. If $L = 2H$, and a body fall from a state of rest till it has acquired a velocity of 30 feet per second,

the space described is $\frac{2}{1} \times \frac{900}{64\frac{1}{3}} = 27.97$ feet, nearly.

Ex. 2. If a body fall 12 feet from a state of rest down this plane, to find the velocity acquired.

Since $S = \frac{L}{H} \times \frac{V^2}{4m}$, we have $V^2 = 4mS \times \frac{H}{L} =$ (in this case) $64\frac{1}{3} \times 12 \times \frac{1}{2} = 386$; hence, $V = 19.6$ feet per second, nearly.

Cor. 7. In the same manner, if a body be acted upon by any uniform force, which is to the force of gravity as $F : 1$, and V represent the velocity generated, T the time in seconds, S the space described, in feet, reckoned from the beginning of the motion, then $V = 2mFT$; $S = mFT^2$; and $V^2 = 4mFS$.

Prop. LXII.

(256.) *The velocity which a body acquires in falling down the whole length of an inclined plane, varies as the square root of the perpendicular height of the plane*.*

In general, when the force is uniform, $V^2 \propto FS$ (Art. 243.); in this case, $F \propto \frac{H}{L}$, and $S = L$, by the supposition; therefore, $V^2 \propto \frac{H}{L} \times L \propto H$; and $V \propto \sqrt{H}$ (*Alg.* Art. 202†.).

(257.) Cor. 1. When the heights of two inclined planes are equal, the velocities acquired in falling down their whole lengths are equal.

* Bodies, in this, and the subsequent Propositions, are supposed to fall from a state of rest.

† See also Cor. 6. Ex. 2. of the last Proposition.

(258.) Cor. 2. The velocity which a body acquires in falling down the length of an inclined plane is equal to the velocity which it would acquire in falling down it's perpendicular height.

Prop. LXIII.

(259.) *The time of a body's descent down the whole length of an inclined plane, varies as the length directly, and as the square root of the perpendicular height inversely.*

In general, $S \propto TV$ (Art. 239.); therefore, $T \propto \frac{S}{V}$, and in this case, $V \propto \sqrt{H}$ (Art. 256.); consequently, $T \propto \frac{S}{\sqrt{H}} \propto \frac{L}{\sqrt{H}}$.

(260.) Cor. 1. If the height, or the last acquired velocity, be given, $T \propto L$.

(261.) Cor. 2. If the inclination be given, or $H \propto L$, then $T^2 \propto \frac{L^2}{L} \propto L$, and $T \propto \sqrt{L}$. That is, the times of descent, down planes equally inclined to the horizon, vary as the square roots of their lengths.

(262.) Cor. 3. The time of descent down an inclined plane, is to the time of falling down it's perpendicular height, as the length of the plane, to it's height.

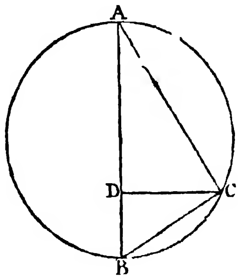
Prop. LXIV.*

(263.) *If chords be drawn in a circle from the extremity of that diameter which is perpendicular to the horizon, the velocities which bodies acquire by falling down them are proportional to their lengths; and the times of descent are equal.*

Let ACB be the circle, AB it's diameter perpen-

* See also Art. 254. Ex. 2.

dicular to the horizon; *BC* a chord drawn from the extremity *B* of the diameter; join *AC*, and draw *CD*



perpendicular to *AB*, or parallel to the horizon. Then *CB* may be considered as an inclined plane whose perpendicular height is *DB*, and the velocity acquired in falling down it varies as $\sqrt{DB}$ (Art. 256.). Now, from the similar triangles *DBC*, *ABC*, $DB : CB :: CB : AB$; therefore, $DB = \frac{CB^2}{AB}$, and $\sqrt{DB} = \frac{CB}{\sqrt{AB}}$; consequently, $V \propto \frac{CB}{\sqrt{AB}}$, and *AB* is invariable; therefore $V \propto CB$.

Again, $T \propto \frac{S}{V}$ (Art. 240.), and in this case, *CB*, which is the space described, has been proved to be proportional to the velocity acquired; therefore $T \propto \frac{CB}{CB}$, or the time of descent is invariable.

(264.) COR. 1. The time of descent down any chord *CB*, is equal to the time of descent down the diameter *AB*.

(265.) COR. 2. In the same manner, the time of descent down *AC* is equal to the time of descent down

AB; therefore the time of descent down *AC* is equal to the time of descent down *CB*.

(266.) Cor. 3. The times of descent down the chords thus drawn, in different circles, are proportional to the square roots of the diameters.

For, the times of descent down the chords are equal to the times of descent down the diameters which are perpendicular to the horizon; and these times vary as the square roots of the diameters. (See Art. 234.)

(267.) When a body falls freely by the force of gravity, every particle in it is equally accelerated; that is, every particle descends towards the horizon with the same velocity; in this descent, therefore, no rotation will be given to the body. The same may be said when a body descends along a perfectly smooth inclined plane, if that part of the force which acts in a direction perpendicular to the plane (Art. 249.), be supported; that is, if a perpendicular to the plane, drawn from the center of gravity of the body, cut the plane in a point which is in contact with the body. If this part of the force be not sustained by the plane, the body will partly roll and partly slide, till this force is sustained; and afterwards the body will wholly slide. When the lateral motion is entirely prevented by the adhesion of the body to the plane, we have before seen on what supposition the body will roll (Art. 186.); if the adhesion be not sufficient to prevent all lateral motion, this body will partly slide and partly roll; and to estimate the space described, the time of it's motion, or the velocity acquired, we must have recourse to other principles than those above laid down.

on this subject the Reader may consult Professor VINCE's *Plan of a Course of Lectures*, p. 39.

(268.) When a body falls freely by the force of gravity, or descends along a perfectly smooth inclined plane, the accelerating force is the same, whatever be the weight of the body (Arts. 248, 249.); consequently, the moving force, on either supposition, is proportional to the quantity of matter moved. In all cases, the accelerating force varies as the moving force directly and the quantity of matter inversely (Art. 24.); and when the moving force and quantity of matter moved are invariable, the accelerating force is uniform, and it's effects may be estimated by the rules laid down in the first part of this section.

Ex. If two bodies, whose weights are P and Q, be connected by a string, and hung over a fixed pulley, to find how far the heavier P will descend in T''.

The moving force of gravity is proportional to the weight; if therefore P be taken to represent the moving force of the former body when it descends freely, Q will represent the moving force of the latter, and $P-Q$ will represent the moving force when the bodies are connected and oppose each other's motion; hence, neglecting the inertia of the string and pulley, the accelerating force of gravity : the accelerating force in this case :: $\frac{P}{P} : \frac{P-Q}{P+Q} :: 1 : \frac{P-Q}{P+Q}$; and, since $FT^2 \propto S$, $1 \times 1^2 : \frac{P-Q}{P+Q} \times T^2 :: 16\frac{1}{12} : 16\frac{1}{12} \times \frac{P-Q}{P+Q} \times T^2$, the space required.

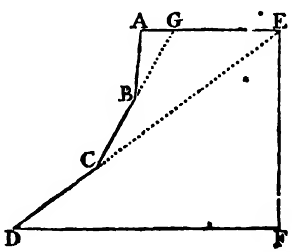
SECTION VIII.

ON THE OSCILLATIONS OF BODIES IN CYCLOIDS AND IN SMALL CIRCULAR ARCS.

Prop. LXV.

(269.) *If a body descend down a system of inclined planes, the velocity acquired, on the supposition that no motion is lost in passing from one plane to another, is equal to that which would be acquired in falling through the perpendicular height of the system.*

Let *ABCD* be the system of planes; draw *AE*,



DF, parallel to the horizon; produce *CB*, *DC*, till

they meet AE in G and E; and draw EF perpendicular to DF. Then the velocity acquired by a body in falling from A to B, is equal to that which it would acquire in falling from G to B, because the planes AB, GB, have the same perpendicular height (Art. 257.); and since, by the supposition, no velocity is lost in passing from one plane to another, the body will begin to descend down BC with the same velocity, whether it fall down AB or GB; consequently, the velocity acquired at C will be the same on either supposition. Also, the velocity acquired at C is equal to that which would be acquired in falling down EC (Art. 257.); and no velocity being lost at C, the body will begin to descend down CD with the same velocity, whether it fall from A through B and C to D, or from E to D; and the velocity acquired in falling down ED is equal to the velocity acquired in falling through the perpendicular height EF (Art. 258.); therefore, the velocity acquired in falling down the whole system, is equal to the velocity acquired in falling through the perpendicular height of the system.

Prop. LXVI.

(270.) *If a body fall from a state of rest down a curve surface which is perfectly smooth, the velocity acquired is equal to that which would be acquired in falling from rest through the same perpendicular height.*

When a body passes from one plane AB to another BC, the whole velocity : the quantity by which the velocity is diminished :: radius : the versed sine of the $\angle ABG$ (Art. 221.); when, therefore, the angle ABG

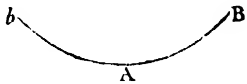
is diminished without limit, the velocity lost is diminished without limit; and if the lengths of the planes, as well as their angles of inclination ABG, BCE, be continually diminished, the system approximates to a curve, as it's limit, in which no velocity is lost; consequently, the whole velocity acquired is equal to that which a body would acquire in falling through the same perpendicular altitude (Art. 269*.).

(271.) Cor. 1. If a body be projected up a curve, the perpendicular height to which it will rise is equal to that through which it must fall to acquire the velocity of projection.

* When the chord of an arc is diminished without limit with respect to the diameter, the versed sine is diminished without limit with respect to the chord; because, the diameter : the chord :: the chord : the versed sine; hence, the ratio of the diameter to the versed sine, and consequently, the ratio of the radius to the versed sine, is, in this case, indefinitely greater than the ratio of the diameter to the chord. Let BC be one of the evanescent planes, V the velocity of the descending body at B, $V+v$ it's velocity at C; produce CB to G, and let GB be the space through which the body must descend to acquire the velocity V; then, $V : V+v :: \sqrt{GB} : \sqrt{GB+BC}$; and when $GB : BC$ :: the radius : an evanescent chord, $V : V+v :: GB : GB+\frac{BC}{2}$ (see Note, p. 126.); therefore, $V : v :: GB : \frac{BC}{2} :: 2GB : BC$. Also, V : the velocity lost at B :: radius : the versed sine of the angle ABG. Hence it follows, that the ratio of V to the velocity lost at B, is indefinitely greater than the ratio of V to the velocity acquired in the descent from B to C; and consequently, the velocity lost at B is indefinitely less than the velocity acquired in the descent from B to C; in the same manner, the velocity lost at any other plane is indefinitely less than the velocity acquired in the descent down that plane; therefore, the velocity lost in the whole descent is indefinitely less than the whole velocity acquired.

For the body in it's ascent will be retarded by the same degrees that it was accelerated in it's descent.

(272.) Cor. 2. If BAb be a curve in which the lowest point is A, and the parts AB, Ab, are similar and equal, a body in falling down BA will acquire a



velocity which will carry it to b; and since the velocities at all equal altitudes in the ascent and descent are equal, the whole time of the ascent will be equal to the time of descent.

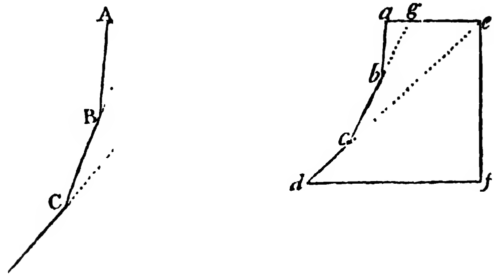
(273.) Cor. 3. The same proposition is true, if the body be retained in the curve by a string which is in every point perpendicular to it. For the string will now sustain that part of the weight which was before sustained by the curve (Art. 117.).

Prop. LXVII.

(274.) *The times of descent down similar systems of inclined planes, similarly situated, are as the square roots of their lengths, on the supposition that no velocity is lost in passing from one plane to another.*

Let $ABCD$, $abcd$, be two similar systems of inclined planes, similarly situated; that is, let $AB : ab :: BC : bc :: CD : cd$; the angles ABC, BCD, respectively equal to the angles abc, bcd; and the planes AB, ab, equally inclined to the horizon. Complete the figures as in the last Proposition; then, since $AB : ab ::$

$BC : bc :: CD : cd$, we have, $AB : ab :: AB+BC+CD : ab+bc+cd$ (*Alg.* Art. 183.); and, $\sqrt{AB} : \sqrt{ab} :: \sqrt{AB+BC+CD} : \sqrt{ab+bc+cd}$. Also, since the angles ABC, abc, are equal, their supplements, the angles ABG, abg, are equal; and the angles of inclination to the horizon BAG, bag, are equal; therefore, the triangles ABG, abg, are similar, and $AB : BG :: ab : bg$; alter. $AB : ab :: BG : bg :: BC :$



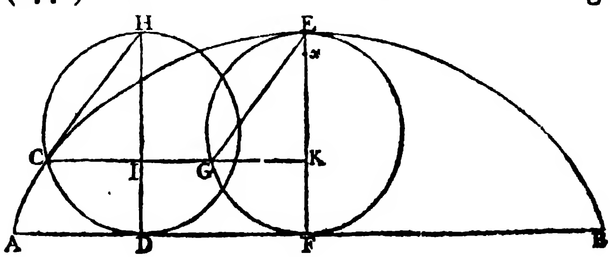
bc; consequently, $BG : bg :: BG+BC\ (GC) : bg+bc\ (gc) :: AB : ab$. In the same manner, $ED : ea :: AB : ab$. Then, because the planes AB, ab, are equally inclined to the horizon, the time of descent down AB : the time down $ab :: \sqrt{AB} : \sqrt{ab}$ (Art. (261.); and if the bodies fall down GC, gc, the time down GC : the time down $gc :: \sqrt{GC} : \sqrt{gc} :: \sqrt{AB} : \sqrt{ab}$; also, the time down GB : the time down $gb :: \sqrt{GB} : \sqrt{gb} :: \sqrt{AB} : \sqrt{ab}$; hence the whole time down GC : the whole time down gc :: the time down GB : the time down gb; therefore, the remainder, the time down BC : the remainder, the time down bc, in the same ratio, or as $\sqrt{AB} : \sqrt{ab}$ (Euc. 19. v.); and since, by the supposition, no mo-

tion is lost in passing from one plane to another, the times of descent down BC and bc are the same, whether the bodies descend from A and a, or from G and g; consequently, when the bodies descend down the systems, the time down BC : the time down bc :: $\sqrt{AB} : \sqrt{ab}$. In the same manner it may be shewn that the time down CD : the time down cd :: $\sqrt{AB} : \sqrt{ab}$. Hence, the time down AB : the time down ab :: the time down BC : the time down bc :: the time down CD : the time down cd; therefore, the time down $AB+BC+CD$: the time down $ab+bc+cd$ :: the time down AB : the time down ab :: $\sqrt{AB} : \sqrt{ab}$ (*Algebra*, Art. 183.) :: $\sqrt{AB+BC+CD} : \sqrt{ab+bc+cd}$.

(275.) COR. 1. If the lengths of the planes, and their angles of inclination ABG, ACE, &c. be continually diminished, the limits, to which these systems approximate, are similar curves, similarly situated, in which no velocity is lost (Art. 270.); hence, the whole times of descent down these curves will be as the square roots of their lengths.

(276.) COR. 2. The times of descent down similar circular arcs, similarly situated, are as the square roots of the arcs, or as the square roots of their radii.

(277.) DEF. If a circle be made to *roll* in a given



plane upon a straight line AB, the point C in the circumference, which was in contact with AB at the beginning of the motion, will, in a revolution of the circle, describe a curve $ACEB$ called a *cycloid.*

The line AB is called the *base* of the cycloid.

The circle HCD is called the *generating circle.*

The line FE, which is drawn bisecting AB at right angles, and produced till it meets the curve in E, is called the *axis*, and the point E, the *vertex*, of the cycloid.

(278.) Cor. 1. The base AB is equal to the circumference of the generating circle; and AF to half the circumference.

(279.) Cor. 2. The axis FE is equal to the diameter of the generating circle.

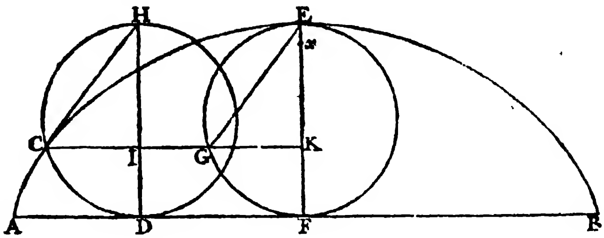
When the generating circle comes to F, draw the diameter Fx, which will be perpendicular to AB (Euc. 18. iii.); and because the circle has completed half a revolution, x is the generating point; that is, x is a point in the cycloid, or x coincides with E.

Prop. LXVIII.

(280.) *If a line* CGK, *drawn from a point* C *in the cycloid, parallel to the base* AB, *meet the generating circle, described upon the axis, in* G, *the circular arc* EG *is equal to the right line* CG.

Let the generating circle HCD touch the base in D when the generating point is at C; draw DH perpendicular to AB, and it will be the diameter of the circle HCD (Euc. 19. iii.), and therefore equal to FE; join CH, GE; and since $DH=FE$, and $DI=FK$ (Euc. 34. i.) the remainders IH and KE are equal; con-

sequently, CI, which is a mean proportional between



HI and ID (Euc. Cor. 8. vi.), is equal to KG, which is a mean proportional between EK and KF; to each of these equals add IG, and $CG=IK$. Also, CH, which is a mean proportional between IH and HD, is equal to GE, which is a mean proportional between EK and EF; therefore, the arc $CH=$the arc GE (Euc. 28. iii.); and since every point in CD has been successively in contact with AD, $CD=AD$, and $HCD=AF$ (Art. 278.); hence, the arc $CH=DF$; therefore, the arc $EG=DF=IK=CG$.

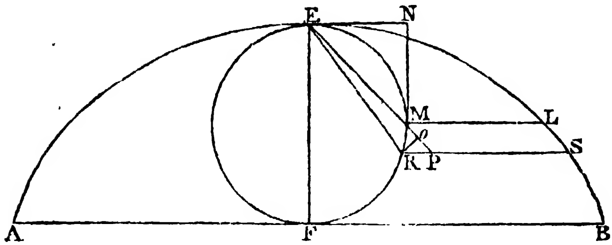
Cor. If the line KGC be always drawn perpendicular to EF the diameter of the circle EGF, and GC be taken equal to the arc EG, the locus of the point C is a cycloid, whose axis is EF.

Prop. LXIX.

(281.) *If a line* LM, *drawn from* L *parallel to the base* AB, *meet the generating circle described upon the axis in* M, *and* EM *be joined, the tangent to the cycloid at the point* L *is parallel to the chord* EM.

Draw SR parallel and indefinitely near to LM; join EM, RM, SL; produce EM till it meets SR in P; draw EN, MN, touching the circle in E and M, and meeting each other in N.

Then, since RM is ultimately in the direction of the



tangent MN (Newt. Lem. 6.), the angles RMP, EMN, are equal; and because EN is parallel to RS (Euc. 18. iii.) the angles MPR, MEN, are equal; therefore, the triangles EMN, RMP, are equi-angular, and $EN : MN :: RP : RM$; and since $EN = NM$, $RP = RM =$ the arc RM (Newt. Lem. 7.). Again, since the arc $EMR = RS$ (Art. 280.), and $RM = RP$, the remainders, the arc EM and the right line PS are equal; also, $ML =$ the arc EM; therefore, $PS = ML$; consequently, SL is equal, and parallel to PM (Euc. 33. i*.); and since SL is ultimately in the direction of the tangent at L (Newt. Lem. 6.), MP, or EM, is parallel to the tangent at L.

(282.) Cor. The tangent to the cycloid at B or A, is perpendicular to AB.

Prop. LXX.

(283.) *The same construction being made, the cycloidal arc* EL *is double of* EM *the corresponding chord of the generating circle described upon the axis.*

* The Proposition may be justly applied, because the difference between LM and SP is evanescent with respect to MP, or LS.

Join ER, and in EP take $Eo=ER$; join Ro. Then, when the arc MR, and consequently the angle MER, is diminished without limit, the sum of the angles ERo, EoR, approximates to two right angles as it's limit; and these angles are equal to each other; therefore, each of them is a right angle; and since the angles RMo, RoM, are respectively equal to RPo, RoP, and Ro is common to the triangles RoM, RPo, $Mo=oP$, and $MP=2Mo$; also, Mo $(ER-EM)$ is the quantity by which the chord EM increases, whilst the cycloidal arc EL increases by LS; and it appears from the demonstration of the last Proposition that $MP=LS=$ arc LS (Newt. Lem. 7.); therefore, the arc $LS=2Mo$; or, the cycloidal arc EL increases twice as fast as the corresponding chord EM; and they begin together at E; consequently, the cycloidal arc EL is double of EM, the corresponding chord of the generating circle.

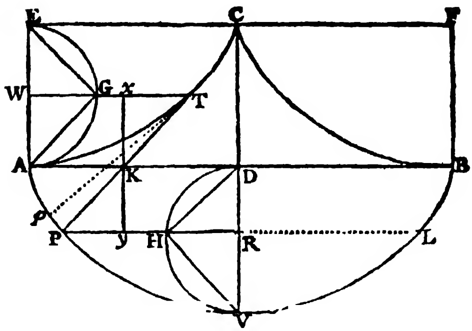
(284.) Cor. The whole semi-cycloidal arc EB is equal to twice the axis EF.

Prop. LXXI.

(285.) *To make a body oscillate in a given cycloid.*

Let AVB be the given cycloid, placed with it's vertex downwards, and it's axis DV perpendicular to the horizon. Produce VD to C, making $DC=VD$; complete the rectangles DE, DF; upon AE describe a semi-circle AGE, and with A as the generating point, and base EC, describe a semi-cycloid ATC; this will pass through the point C, because the semi-circumference $AGE=DHV=AD=EC$; in the same manner,

describe an equal semi-cycloid between C and B.



Then, if a body P be suspended from C by a string whose length is CV or CTA (Art. 284.), and made to vibrate between the cycloidal cheeks CA, CB, it will always be found in the cycloid AVB.

Let the string be brought into the situation CTP, and since it is constantly stretched by the gravity, and the centrifugal force of P, it will be a tangent to the cycloid at the point T where it leaves the curve. From T and P draw TGW, PHR, parallel to AD; join AG, GE, DH, HV; and through K draw xKy perpendicular to TG or PH. Then, since the chord AG is parallel to TP (Art 281.), and TG is parallel to AK, the figure GK is a parallelogram, and $AG=TK$, $GT=AK$; and because the length of the string is equal to CTA, and the part CT is common to the string and the cycloidal arc, $TP=AT=2AG$ (Art. 283.) $=2TK$; or $TK=KP$; hence, the triangles TKx, PKy, are similar and equal, and $Kx=Ky$; also, $Kx=AW$ and $Ky=DR$; therefore $AW=DR$, and $AE=DV$; hence, the arc $AG=$ the arc DH; and the angle

GEA = the angle HVD; or, the angle GAK = the angle KDH (Euc. 32. iii.); consequently, AG is parallel to DH; and therefore, TP is parallel to DH, and the figure $KPHD$ is a parallelogram; hence, $KD = PH$. Again, since the arc $AG = GT$ (Art. 280.) $= AK$, the arc $DH = AK$; and the semi-circumference $DHV = AD$; therefore, the arc $VH = KD = HP$; that is, P is in the cycloid, whose axis is DV, and vertex V (Art. 280.).

(286.) Cor. 1. Since DH is parallel to TP, and VH to the tangent at P, the angle contained between TP and the tangent, or between TP and the curve, is equal to the angle DHV; that is, TP is always perpendicular to the curve.

(287.) Cor. 2. If Pp be an evanescent arc, the perpendiculars to the curve at P and p, ultimately meet in T; and Pp may be considered as a circular arc whose radius is TP.

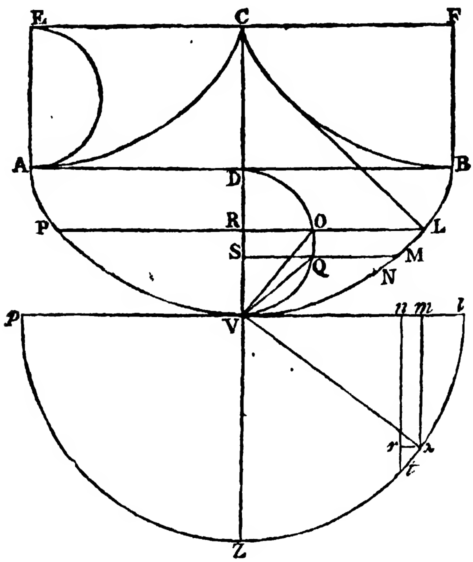
(288.) Cor. 3. An evanescent arc at the vertex of the cycloid may be considered as a circular arc whose radius is CV.

(289.) Def. If a body begin to descend in a curve, from any point, and again ascend till it's velocity is destroyed (Art. 272.), the time in which the motion is performed is called *the time of an oscillation.*

Prop. LXXII.

(290.) *If a body, vibrating in the cycloid* AVB, *begin to descend from* L, *the velocity acquired at any point* M[*] *varies as* $\sqrt{VL^2 - VM^2}$; *or, as the right sine of a circular arc whose radius is equal to* VL, *and versed sine to* LM.

From the points L and M, draw LOR, MQS, at right angles to DV, meeting the circle DOV in O



and Q; join OV, QV; with the radius $Vl = VL$, describe the semi-circle lZp, and take $lm = LM$; draw mx, VZ, at right angles to Vl; and join Vx.

The velocity acquired in the descent from L to M, is equal to the velocity acquired in falling from R to S (Art. 270.); and therefore it varies as $\sqrt{RS}$ (Art. 241.); that is, $\propto \sqrt{RV - SV} \propto \sqrt{DV \times RV - DV \times SV}$ (because DV is invariable), $\propto \sqrt{VO^2 - VQ^2} \propto \sqrt{4VO^2 - 4VQ^2} \propto \sqrt{VL^2 - VM^2}$ (Art. 283.), $\propto \sqrt{Vl^2 - Vm^2} \propto \sqrt{Vx^2 - Vm^2} \propto \sqrt{mx^2} \propto mx$.

(291.) Cor. The velocity at M : the velocity at V :: mx : VZ :: mx : Vx.

Prop. LXXIII.

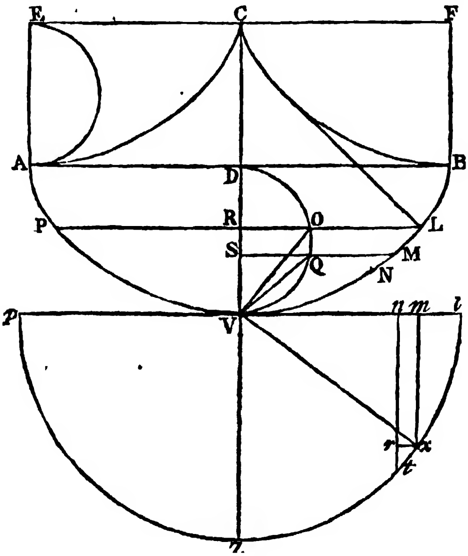
(292.) *The time of an oscillation in the arc* LVP, *is equal to the time in which a body would describe the semi-circumference* lZp, *with the velocity acquired at* V *continued uniform.*

Let MN be a very small arc, and take $mn = MN$; draw nt, xr, respectively parallel to mx and Vl; and suppose a body to describe the circumference lZp with the velocity acquired at V continued uniform. Then, when MN is diminished without limit, the velocity with which it is described : the velocity with which xt is described :: mx : VZ (Art. 291.); therefore, the time of describing MN : the time of describing xt :: $\frac{MN}{mx} : \frac{xt}{VZ} :: \frac{xr}{mx} : \frac{xt}{Vx}$ (Art. 15.) Now, the triangles Vxm, xrt are ultimately similar, and $Vx : mx :: xt : xr$; therefore $\frac{xr}{mx} = \frac{xt}{Vx}$; consequently, the time of descent down MN, is equal to the time of describing the corresponding circular arc xt with the velocity VZ; and the same may be proved of all other corresponding arcs in the cycloid and the circle; therefore the whole time of an oscillation is equal to the time of describing the semi-circumference lZp, with the velocity acquired at V continued uniform.

Prop. LXXIV.

(293.) *The time of an oscillation in a cycloid is to the time of descent down it's axis, as the circumference of a circle to it's diameter.*

If a body fall down the chord OV, the velocity acquired at V is equal to the velocity in the cycloid at V (Art. 270.); and with this velocity continued uniform, the body would describe $2OV$, or VL, or Vl, in the time of descent down OV (Art. 237.); that is, in the time of descent down DV (Art. 264.). It



appears then, that the time of an oscillation is equal to the time of describing lZp with the velocity acquired in the cycloid at V (Art. 292.); and that the time of descent down the axis DV is equal to the time of describing Vl with the same velocity; therefore, the time of an oscillation : the time of descent down the axis :: the time of describing the circumference lZp, with the velocity VZ : the time of describing Vl with the same velocity :: $lZp : Vl$ (Art. 13.) :: $2lZp : 2Vl$ :: the circumference of a circle : it's diameter.

(294.) Cor. 1. The time of an oscillation in a given cycloid, at a given place, is the same, whether the body oscillate in a greater or a smaller arc.

For, the time of an oscillation bears an invariable ratio to the time of descent down the axis, which, in a given cycloid, at a given place, is given.

(295.) Cor. 2. The time of an oscillation in a small circular arc whose radius is CV, is to the time of descent down $\frac{1}{2}CV$, as the circumference of a circle to it's diameter.

For, the time of an oscillation in this circular arc is equal to the time of an oscillation in an equal arc of the cycloid AVB (Art. 288.).

(296.) Cor. 3. The time of an oscillation in a cycloid, or small circular arc, when the force of gravity is given, varies as the square root of the length of the string.

For, the time of an oscillation varies as the time of descent down half the length of the string; that is, as the square root of half the length of the string, or as the square root of it's whole length.

Ex. 1. To compare the times in which two pendulums vibrate, whose lengths are 4 and 9 inches.

Since $T \propto \sqrt{L}$, we have $T : t :: \sqrt{4} : \sqrt{9} :: 2 : 3$.

Ex. 2. If a pendulum, whose length is 39.2 inches, vibrate in one second, in what time will a pendulum vibrate whose length is L inches?

$\sqrt{39.2} : \sqrt{L} :: 1 : T = \sqrt{\dfrac{L}{39.2}}$, the time required, in seconds.

Ex. 3. To compare the lengths of two pendulums, whose times of oscillation are as 1 to 3.

Since $T \propto \sqrt{L}$, $T^2 \propto L$; therefore, $1 : 9 :: L : l$.

(297.) Cor. 4. The number of oscillations, which a pendulum makes in a given time, at a given place, varies inversely as the square root of it's length.

Let n be the number of oscillations, t the time of one oscillation; then, nt is the whole time, which, by the supposition, is given; therefore, $n \propto \frac{1}{t}$ (*Alg.* Art. 206.), and $t \propto \sqrt{L}$; consequently, $n \propto \frac{1}{\sqrt{L}}$.

Ex. 1. If a pendulum, whose lengh is 39.2 inches, vibrate seconds, or 60 times in a minute, how often will a pendulum whose length is 10 inches vibrate in the same time?

Since $n \propto \frac{1}{\sqrt{L}}$, we have $\sqrt{10} : \sqrt{39.2} :: 60 : 60 \times \sqrt{39.2} = 118.8$, nearly, the number of oscillations required.

Ex. 2. If a pendulum, whose length is 39.2 inches, vibrate seconds, to find the length of a pendulum which will vibrate double seconds, or 30 times in a minute.

Since $n \propto \frac{1}{\sqrt{L}}$, we have, $L \propto \frac{1}{n^2}$; and in this case, $\overline{30}|^2 : \overline{60}|^2 :: 39.2 : L = 4 \times 39.2 = 156.8$ inches, the length required.

. Ex. 3. To find how much the pendulum of a clock, which loses one second in a minute, ought to be shortened.

Since the pendulum vibrates 59 times, whilst a pendulum of 39.2 inches vibrates 60 times, it's length may be found as in the last example; $\overline{59}|^2 : \overline{60}|^2 ::$ 39.2 : 40.5, it's length; and it ought to be 39.2 inches; therefore, $40.5 - 39.2$, or 1.3 inches, is the quantity by which it ought to be shortened, in order that it may vibrate seconds.

(298.) Cor. 5. If the force of gravity be not given, the time of an oscillation varies as the square root of the length of the pendulum directly, and as the square root of the force of gravity inversely.

For, the time of an oscillation varies as the time of descent down half the length of the string; and in general, the time of descent through any space $\propto \sqrt{\frac{S}{F}}$ (Art. 241.); in this case, $S = \frac{1}{2} L$; therefore $S \propto L$, and the time of descent $\propto \sqrt{\frac{L}{F}}$; hence T, the time of an oscillation, $\propto \sqrt{\frac{L}{F}}$.

(299.) Cor. 6. If the length of the pendulum be given, $T \propto \frac{1}{\sqrt{F}}$; and $F \propto \frac{1}{T^2}$.

The time in which a given pendulum vibrates, increases as it is carried from a greater latitude on the earth's surface to a less; therefore, the force of gravity decreases as the latitude decreases.

(300.) Cor. 7. The force of gravity at the equator : the force of gravity at any proposed latitude :: the

length of a pendulum which vibrates seconds at the equator : the length of a pendulum which vibrates seconds at the proposed latitude.

For, $T \propto \sqrt{\frac{L}{F}}$; if therefore T be given, $\sqrt{F} \propto \sqrt{L}$, or $F \propto L$.

(301.) Cor. 8. If the chord BV* be drawn, the time of descent down the cycloidal arc BV : the time of descent down the chord :: DB : BV.

For, the time of descent down the arc BV is equal to half the time of an oscillation (Art. 272); therefore, the time of descent down the arc BV ; the time of descent down DV :: half the circumference of a circle : it's diameter :: DB : DV; also, the time of descent down DV : the time of descent down the chord BV :: DV : BV; therefore, *ex æquo*, the time of descent down the arc BV : the time of descent down the chord :: DB : BV.

Prop. LXXV.

(302.) *The space through which a body falls by the force of gravity in the time of an oscillation in a cycloid, or small circular arc, is to half the length of the pendulum, as the square of the circumference of a circle to the square of it's diameter.*

The spaces through which bodies fall by the action of the same uniform force are as the squares of the times (Art. 241.); and since the time of an oscillation : the time of descent down half the length of the pendulum :: the circumference of a circle : it's diameter, the space fallen through in the time of an oscillation :

* The line BV is wanting in the figure.

half the length of the pendulum :: the square of the circumference of a circle : the square of it's diameter.

Ex. To find how far a body will fall by the force of gravity in one second, where the length of the pendulum, which vibrates seconds, is 39.2 inches.

The circumference of a circle : it's diameter :: 3.14159 : 1; consequently, the space fallen through in one second : $\frac{39.2}{2}$ :: $\overline{3.14159}|^2$: 1^2; hence, the space fallen through is $19.6 \times \overline{3.14159}|^2 = 193$ inches, or $16\frac{1}{12}$ feet, nearly.

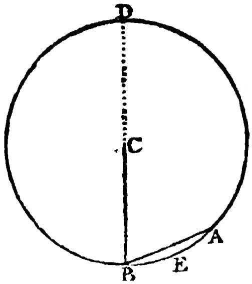
If the arc, in which a body oscillates, be diminished, the effect of the air's resistance is diminished; and when the arc is very small, this resistance does not sensibly affect the time of an oscillation. By observing, therefore, the length of a pendulum which vibrates seconds in very small arcs, we determine the space through which a body would fall *in vacuo* in one second, with sufficient accuracy for all practical purposes.

Prop. LXXVI.

(303.) *The time of descent to the lowest point in a small circular arc is to the time of descent down it's chord, as the circumference of a circle to four times the diameter.*

Let AB be the arc, C it's center, BD the diameter perpendicular to the horizon; T the time of descent down the arc AEB, t the time of descent down the chord, C the circumference of a circle, D it's diameter. Then, $2T$ is the time of an oscillation of the pendulum

CB; therefore, $2T$: the time down $\frac{CB}{2}$ :: C : D (Art. 295.); and the time down $\frac{CB}{2}$: the time down



DB, or AB, :: 1 : 2 (Art. 241.); therefore, $2T$: the time down AB (t) :: C : $2D$, and T : t :: C : $4D$.

Prop. LXXVII.

(304.) *The force, which accelerates or retards a body's motion in a cycloid, varies as the arc intercepted between the body and the lowest point.*

Let DV represent the whole force of gravity, (See Fig. in p. 166.) from P draw PH parallel to AD meeting the circle DHV in H; join DH, HV.

Then, the whole force DV, which acts upon the body at P, may be resolved into the two DH, HV; of which DH is in the direction of the string, and therefore neither accelerates nor retards the motion of P; and HV is in the direction of the tangent at P (Art. 281.), and therefore wholly employed in accelerating or retarding the motion in the curve; consequently, the force of gravity : the accelerating force :: DV : HV; and since the force of gravity, and DV, are invariable, the accelerating force $\propto HV \propto 2HV \propto PV$ (Art. 283.).

(305.) Cor. 1. If a body move in any line, and be acted upon by a force which varies as the distance from the lowest point, the motion of this body will be similar to the motion of a body oscillating in a cycloid.

For, if an arc, measured from the vertex of a cycloid, be taken equal to the line, and the accelerating forces, in the line and the cycloid, at these equal distances from the lowest points, be equal, they will always be equal, because they vary according to the same law; and the bodies, being impelled by equal forces, will be equally accelerated, and describe equal spaces in any given time.

(306.) Cor. 2. The time of descent to the lowest point in the line will always be the same, from whatever place the body begins to fall.

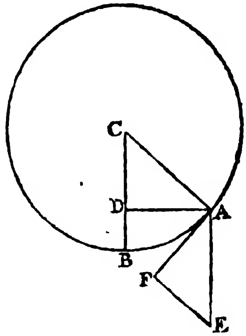
(307.) Cor. 3. If the distance of the body from the lowest point at the beginning of the descent, be made the radius, the velocity acquired will be represented by the right sine, and the time by the arc, whose versed sine is the space fallen through.

Prop. LXXVIII.

(308.) *If a body vibrate in a circular arc, the force which accelerates or retards it's motion varies as the sine of it's distance from the lowest point.*

Let a body oscillate in a circular arc whose radius is AC; from the center C, and A the place of the body, draw CB, AE perpendicular to the horizon, and take AE to represent the force of gravity; draw AD perpendicular to CB, and EF perpendi-

cular to AF, which is a tangent to the circle at A.



Then, the force AE is equivalent to the two AF, FE; of which FE is perpendicular to the tangent AF, or in the direction of the radius CA, and can neither accelerate nor retard the motion of the body; the other, AF, is in the direction of the tangent, and is wholly employed in accelerating or retarding the body's motion; therefore, the force of gravity : the accelerating force :: $AE : AF$, that is, from the similar triangles AEF, CAD, :: $CA : AD$; and consequently, the accelerating force $= \frac{\text{gravity} \times AD}{AC}$; in which expression, gravity and the radius AC are invariable; therefore, the accelerating force varies as AD.

(309.) Cor. 1. If the accelerating force were proportional to the arc, the oscillations, whether in greater or smaller arcs, would be performed in equal times (Art. 306.); but, since the sine does not increase as fast as the arc, the force in the greater arc is less than that which would be sufficient to make the time of oscillation equal to the time in the smaller arc;

therefore, the time of oscillation in the greater circular arc is greater than in the less.

(310.) Cor. 2. Call F the force in the direction AE; then, since AC is invariable, the accelerating force in the curve $\propto F \times AD$; and if $F \times AD \propto AB$, or $F \propto \frac{AB}{AD}$, the accelerating force varies as AB, and the times of oscillation in different arcs are equal (Art. 306.).

SCHOLIUM.

(311.) In this Section we have considered the vibrations of a simple pendulum only, or of a single particle of matter, suspended by a string, the gravity of which is neglected. The Propositions are indeed applicable in practice, when the diameter of the body is small with respect to the length of the string by which it is suspended, and the weight of the string inconsiderable when compared with the weight of the body. That the conclusions are not strictly true in this case, is evident from the consideration that two particles of matter, at different distances from the axis of suspension, do not vibrate in the same time (Art. 296.); and consequently, that when they are connected together, they affect each other's motion; thus, the time of vibration of the two particles when united, is different from the time in which either would vibrate alone.

The method of determining the time of vibration of a compound pendulum, the Reader will find in the *Principles of Fluxions,* Art. 63; to which place he is also referred for the investigation of the rules for determining the centers of *Gyration* and *Percussion;* questions properly belonging to Mechanics, but inserted in that part of the Work, because the rules cannot easily be applied to the determination of those points, even in the most simple cases, without the assistance of the fluxional calculus.

(312.) To avoid the introduction of analytical demonstrations in subjects professedly geometrical,

Sir I. NEWTON and other Writers, have had recourse to indefinitely small or evanescent increments, which continually approximate to the true increments of the quantities whose finite values are required. This method may be applied with success in all cases where the *difference* between the *assumed* and the *true* increments continually decreases, and at length vanishes, with respect to the increments themselves; or, which amounts to the same thing, when the ratio which the sum of the differences bears to one of the increments, does not exceed a finite ratio: for, by observing the *limit* to which the sum of the assumed increments approaches, when their number is increased and their magnitudes are diminished *in infinitum*, it is evident that the sum of the real increments is obtained. In the same manner, when there are two ranks of quantities, in which the assumed increments continually approximate to the real increments, as in the former instance, and the limiting ratio of the sums of the assumed increments in these cases, when their numbers are increased and their magnitudes diminished without limit, is obtained, the exact ratio of the quantities themselves is obtained. These propositions are laid down by Sir. I. NEWTON in the first Section of the Principia, Lem. 3d and 4th, and the same mode of reasoning has been applied in Art. 292, to compare the time of an oscillation in the cycloid BVA, with the time of describing the arc lZp with the velocity acquired at V continued uniform. In this Art. it is supposed that the time of describing MN, with the *uniform* velocity mx, is the increment of the former time, and that the time of describing xt, the

side of a triangle *similar* to Vxm, with the velocity VZ, is the increment of the latter; these *assumed* increments, it is manifest, differ from the true increments of the times under consideration; but when they are diminished without limit, they differ from them by quantities which are evanescent with respect to the whole increments, and therefore by determining the limiting ratio of the sums of the assumed increments, we obtain the ratio of the actual times of describing the corresponding arcs.

SECTION IX.

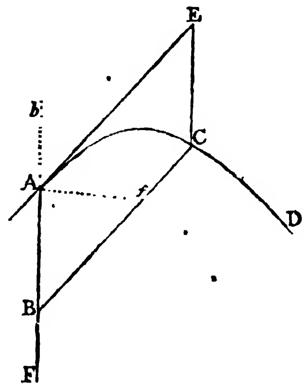
ON THE MOTION OF PROJECTILES.

Prop. LXXIX.

(313.) *A body projected in any direction, not perpendicular to the horizon, will describe a parabola, on supposition that the force of gravity is uniform, and acts in parallel lines, and that the motion is not affected by the resistance of the air.*

Let a body be projected from A in the direction AE, from which point draw ABF perpendicular to the horizon; also, let AE be the space over which the velocity of projection would carry the body in any time, T, and AB the space through which the force of gravity would cause it to descend in the same time; complete the parallelogram AC; then, in consequence of the two motions, the body will be found in C at the end of that time. For, the motion in that direction

AE can neither accelerate nor retard the approach of the body to the line BC (Art. 29.); therefore, at the end of the time T, the body will be in the line BC;



and, by the same mode of reasoning, it appears that it will be in the line EC at the same time; consequently, it will be at C, the point of their intersection, at the end of the time T. Now, since AE is the space which would be described in the time T, with the velocity of projection continued uniform, $AE \propto T$ (Art. 13.); and $BC = AE$; therefore $BC \propto T$, and $BC^2 \propto T^2$. Also, since AB is the space through which the body would fall by the force of gravity in the time T, $AB \propto T^2$ (Art. 241.); hence, $AB \propto BC^2$; and this is the property of a parabola, in which AF is a diameter, and BC an ordinate to the abscissa AB.

(314.) Cor. 1. The axis, and all the diameters of the parabola described, being parallel to AF, are perpendicular to the horizon.

(315.) Cor. 2. The direction of projection AE is parallel to the ordinate BC, and therefore it is a tangent to the curve at A.

(316.) Cor. 3. The time in which a body describes the arc AC is equal to the time in which it would fall from A to B by the force of gravity, or describe AE with the first velocity continued uniform.

(317.) Cor. 4. If the $\angle EAf$ be made equal to the $\angle EAb$, the line Af will pass through the focus of the parabola described.

(318.) Cor. 5. The body will always be found in the plane ACB which passes through the direction of projection and the perpendicular to the horizon.

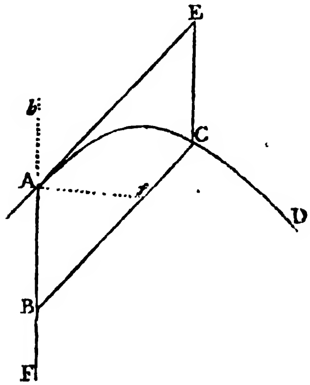
(319.) Cor. 6. If the motion in the direction AE be produced by the action of an uniform force, $AE \propto T^2 \propto AB$; or $AB \propto BC$; therefore, the locus of the point C is a right line.

Prop. LXXX.

(320.) *The velocity of the projectile, at any point in the parabola, is such as would be acquired in falling through one fourth part of the parameter belonging to that point.*

Let AB be the space through which a body must fall by the force of gravity to acquire the velocity of the projectile at A; and AE the space described with that velocity continued uniform, in the time of falling through AB; then $2AB = AE$ (Art. 236.); and, completing the parallelogram AC, $2AB = BC$; hence, $4AB^2 = BC^2$. Also, since C is a point in the parabola,

and BC an ordinate to the abscissa AB (Art. 313.),



if P be the parameter belonging to the point A, $P \times AB = BC^2 = 4AB^2$; therefore $AB = \frac{1}{4}P$.

(321.) Cor. 1. If the velocity at A be given, the parameter at that point is the same, whatever be the direction of the body's motion.

(322.) Cor. 2. If AB be the space through which a body must fall to acquire the velocity at A, and a circle be described from the center A with the radius AB, the focus of the parabola described will lie in the circumference of this circle, whatever be the direction of projection; since the distance of any point in the parabola from the focus, is one fourth part of the parameter belonging to that point.

(323.) Cor. 3. The velocity in a parabola varies as the square root of the parameter.

Since the velocity is such as would be acquired in falling through $\frac{1}{4}P$, it varies as $\sqrt{\frac{1}{4}P}$ (Art. 241.), or as $\sqrt{P}$.

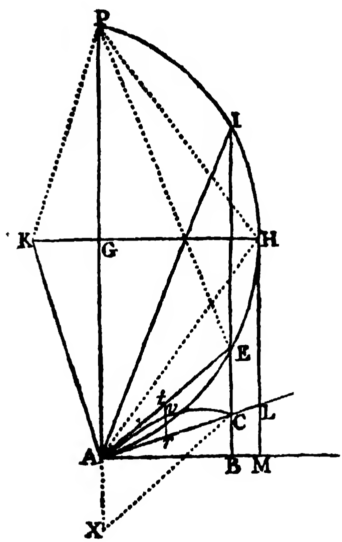
(324.) Cor. 4. The velocity is the least in the vertex of a parabola; and at equal distances from the vertex the velocities are equal.

For, the parameter belonging to the vertex is the least; and the parameters at equal distances from the vertex are equal.

Prop. LXXXI.

(325.) *To find the direction in which a body must be projected from a given point, with a given velocity, to hit a given mark.*

Let A be the point from which the body is to be



projected, C, the given mark; join AC, and from A

draw AB parallel, and AP perpendicular to the horizon; take AP equal to four times the space through which a body must fall to acquire the given velocity of projection, (determined by Art. 248, CASE 3.); then will AP be the parameter belonging to the point A of the parabola described (Art. 320.). Draw AK perpendicular to AC; bisect PA in G, and draw KGH perpendicular to AP, meeting AK in K; join KP; then the triangles KGP, KGA, being similar and equal, $KP = KA$. From K as a center, with the radius KA, or KP, describe a circle AHP, cutting KGH in H; through C draw CEI parallel to AP, and cutting the circle in E and I; join AE, AI; and if a body be projected, with the given velocity, in the direction AE or AI, it will hit the mark C.

Let the body be projected in the direction AE; join PE, and complete the parallelogram $AECX$; then AX is a diameter of the parabola described; and XC, which is parallel to the tangent AE, is in the direction of an ordinate to the abscissa AX; if then XC be the length of the ordinate to this abscissa, C is a point in the parabola. Now, since the angles AEC, EAP, are alternate angles, and the angle EAC = the angle EPA, because AC is a tangent to the circle at A (EUC. 16. iii.), the triangles EPA, EAC, are similar; and $AP : AE :: AE : EC$; or by substituting for AE and EC their equals XC and AX, $AP : XC :: XC : AX$; that is, XC is a mean proportional between the parameter and the abscissa; and therefore it is the ordinate belonging to that abscissa; hence, C is a point in the parabola which the body describes.

In the same manner it may be shewn that, if the

body be projected with the same velocity in the direction AI, it will hit the mark C.

(326.) Cor. 1. Join AH, HP; then the angle HAP = the angle HPA = the angle HAC.

(327.) Cor. 2. Because KH is drawn through the center of the circle perpendicular to the chord EI, it bisects it, and consequently it bisects the arc IHE (Euc. 30. iii.); therefore, the angle IAH = the angle HAE. That is, the two directions AE, AI, make equal angles with AH, which bisects the angle PAC.

(328.) Cor. 3. Draw HLM touching the circle in H; then, when the point C coincides with L, the two directions AE, AI, coincide with AH.

(329.) Cor. 4. If the point C be taken in the plane AL, beyond L, the line CEI will not meet the circle. In this case, the velocity of projection is not sufficient to carry the body to the distance AC.

(330.) Cor. 5. Bisect AC in r, and draw tvr parallel to AP; then tvr is in the direction of a diameter, to which AC is a double ordinate. Also, rt is the sub-tangent; and if rt be bisected in v, this point is in the parabola.

(331.) Cor. 6. A tangent to the parabola at v is parallel to the ordinate AC; therefore, v is the point in the parabola which is at the greatest distance from AC*.

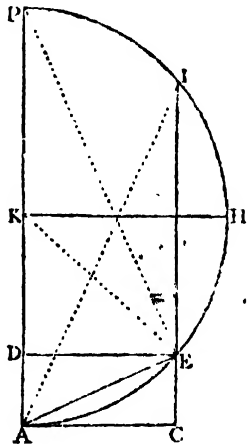
(332.) Cor. 7. The greatest height of the projectile above the plane, measured in the direction of gravity, is $\frac{1}{4}EC$. For, $rv = \frac{1}{2}rt$, and $rt = \frac{1}{2}EC$; therefore, $rv = \frac{1}{4}EC$.

* The properties of the parabola here referred to, may be found in any Treatise of Conic Sections.

Prop. LXXXII.

(333.) *Having given the velocity and direction of projection, to find where the body will strike the horizontal plane which passes through the point of projection.*

Let AC (Art. 325.) coincide with the horizontal line AB; then AK coincides with AG; and PHA is



a semi-circle; also, HAC is an angle of 45°, and AE, AI, are equally inclined to AH.

From the last Proposition it appears, that if the velocity of projection be such as would be acquired by falling through $\frac{1}{4}PA$, and AE, or AI, be the direction of projection, the range is AC. From E draw ED parallel to AC, or perpendicular to AP; join EK; then ED, or it's equal AC, is the sine of the $\angle EKA$ to the radius KA; and the $\angle EKA = 2\angle EPA = 2\angle EAC$;

therefore AC is the sine of $2 \angle EAC$ to the radius KA; and the sine of a given angle is proportional to the radius; consequently, rad. : KA :: sin. $2 \angle EAC$: AC; hence, $AC = \frac{\text{sin. } 2 \angle EAC \times KA}{\text{rad.}}$. If V be taken to represent the velocity of projection, P the parameter AP, and $m = 16\frac{1}{12}$, then $AC = \frac{\text{sin. } 2 \angle EAC \times P}{2 \text{ rad.}} = \frac{\text{sin. } 2 \angle EAC \times V^2}{2 \text{ rad.} \times m}$ (Art. 248.)

In the same manner, if AI be the direction of projection, $AC = \frac{\text{sin. } 2 \angle IAC \times P}{2 \text{ rad.}} = \frac{\text{sin. } 2 \angle IAC \times V^2}{2 \text{ rad.} \times m}$.

(334.) Cor. 1. Hence, $AC \propto \text{sin. } 2 \angle EAC \times V^2$.

(335.) Cor. 2. If the velocity of projection be invariable, the horizontal range varies as sin. $2 \angle EAC$.

(336.) Cor. 3. The range is the greatest when sin. $2 \angle EAC$ is the greatest; that is, when the $\angle EAC$ is In this case, $AC = \frac{\text{rad.} \times P}{2 \text{ rad.}} = \frac{1}{2} P$.

(337.) Cor. 4. If the angle EAC be 15°, or 75°, sin. $2 \angle EAC = \frac{1}{2}$ rad. and $AC = \frac{1}{4} P$.

(338.) Cor. 5. If the range and the parameter be given, the angle of elevation may be found.

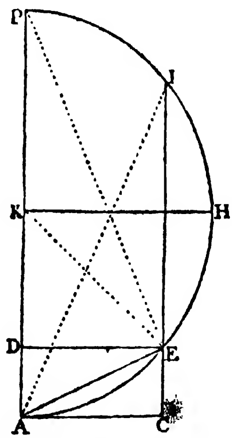
For, $AC = \frac{\text{sin. } 2 \angle EAC \times P}{2 \text{ rad.}}$; therefore, sin. $\angle EAC = \frac{2AC \times \text{rad.}}{P}$.

(339.) Cor. 6. If AC and the angle EAC be known, $P = \frac{2AC \times \text{rad.}}{\text{sin. } 2 \angle EAC}$; and $V^2 = \frac{2 \text{ rad.} \times mAC}{\text{sin. } 2 \angle EAC}$.

Prop. LXXXIII.

(340.) *The same things being given, to find the time of flight.*

The time in which the velocity of projection, V, would be acquired, or the time of descent down $\frac{1}{4}PA$, is $\frac{V}{2m}''$ (Art. 248.); hence, the double of this time, or the time of descent down PA, is $\frac{V}{m}''$ (Art. 241.); which is also the time of descent down EA (Art. 264.) Let T be the time of descent down PA, or EA, t the time of descent down EC, or the time of flight (Art. 316.);



then $T : t :: EA : EC$ (Art. 262.); that is, $T : t ::$ rad. : sin. $\angle EAC$, and $t = \frac{\text{sin. } \angle EAC \times T}{\text{rad.}} = \frac{\text{sin. } \angle EAC \times V}{\text{rad.} \times m}$.

(341.) Cor. 1. If the velocity of projection be invariable, the time of flight $\propto$ sin. $\angle EAC$.

(342.) Cor. 2. Hence, the time of flight is the greatest, when sin. $\angle EAC$ is the greatest. In this case, the time becomes $\frac{\text{rad.} \times T}{\text{rad.}}$, or T; that is, the greatest time of flight is equal to the time of descent down the parameter.

Prop. LXXXIV.

(343.) *The same things being given, to find the greatest height to which the projectile rises above the horizontal plane.*

The greatest height is $\frac{1}{4}EC$, or $\frac{1}{4}AD$; and AD is the versed sine of the $\angle AKE$, or $2\angle EAC$, to the radius AK; consequently, since the versed sine of a given angle varies as the radius, rad. : AK $(\frac{1}{2}P)$ :: the versed sine of $2\angle EAC$: AD. Hence, $AD = \frac{\text{ver. sin. } 2\angle EAC \times P}{2 \text{ rad.}}$; and $\frac{1}{4}AD$, the greatest height,

$$= \frac{\text{ver. sin. } 2\angle EAC \times P}{8 \text{ rad.}} = \frac{\text{ver. sin. } 2\angle EAC \times V^2}{8 \text{ rad.} \times m}$$

(344.) Cor. 1. The greatest height $\propto$ ver. sin. $2\angle EAC \times V^2$; and when V is given, the greatest height $\propto$ ver. sin. $2\angle EAC$.

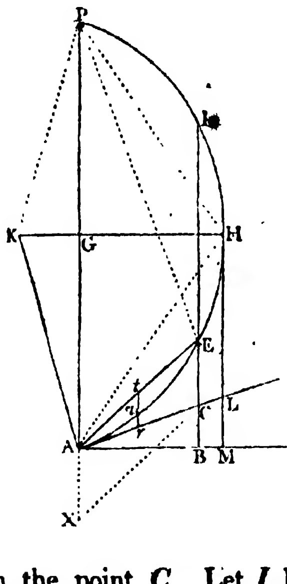
(345.) Cor. 2. The versed sine of an arc varies as the square of the sine of half the arc; therefore the greatest height $\propto \overline{\text{sin. } \angle EAC}|^2 \times V^2$.

(346.) Cor. 3. A body, projected with a given velocity, will rise to the greatest height above the horizontal plane, when the angle of elevation EAC is a right angle. In this case, ver. sin. $2\angle EAC = 2$ rad. and the greatest altitude is $\frac{2 \text{ rad.} \times P}{8 \text{ rad.}} = \frac{P}{4}$.

Prop. LXXXV.

(347.) *The velocity and direction of projection being given, to find where the body will strike a given inclined plane which passes through the point of projection.*

It appears from Art. 325, that if a body be projected from A, in the direction AE, with the velocity acquired in falling down $\frac{1}{4}PA$, it will strike the



plane AC in the point C. Let I be the angle of inclination CAB; E the angle of elevation EAC; Z the angle EAP. Then, in the triangle EAP, $AE : AP :: \sin. \angle EPA : \sin. \angle AEP$; and the $\angle EPA =$ the $\angle EAC = E$; also the $\angle AEP =$ the $\angle ECA =$ the supplement of the $\angle ACB$; hence, AE

$: AP :: \sin. E : \cos. I$; therefore, $AE = \frac{\sin. E \times AP}{\cos. I}$. Again, in the triangle EAC, $AC : AE :: \sin. \angle AEC$ ($\sin. Z$) : $\sin. \angle ACE$ ($\cos. I$); therefore, $AC = \frac{\sin. Z \times AE}{\cos. I}$; and by substituting for AE it's value $\frac{\sin. E \times AP}{\cos. I}$, $AC = \frac{\sin. E \times \sin. Z \times AP}{\overline{\cos. I}|^2} = \frac{\sin. E \times \sin. Z \times V^2}{\overline{\cos. I}|^2 \times m}$.

(348.) Cor. Hence, $AC \propto \frac{\sin. E \times \sin. Z \times V^2}{\overline{\cos. I}|^2}$.

Prop. LXXXVI.

(349.) *The same things being given, to find the time of flight.*

Let T be the time of descent down PA, t the time of descent down EC or the time of flight; then, $T^2 : t^2 :: PA : EC$; and since, in the similar triangles PAE, AEC, $PA : AE :: AE : EC$, we have $PA^2 : AE^2 :: PA : EC :: T^2 : t^2$; and $PA : AE :: T : t$; but $PA : AE :: \sin. \angle PEA : \sin. \angle EPA :: \sin. \angle ECA : \sin. \angle EAC :: \cos. I : \sin. E$; therefore, $T : t :: \cos. I : \sin. E$, and $t = \frac{\sin. E \times T}{\cos. I} = \frac{\sin. E \times V}{\cos. I \times m}$.

(350.) Cor. Hence, $t \propto \frac{\sin. E \times V}{\cos. I}$; and if V be invariable, $t \propto \frac{\sin. E}{\cos. I}$.

Prop. LXXXVII.

(351.) *The same things being given, to find the greatest height of the projectile above the plane* AC, *measured in the direction of gravity.*

The greatest height is $\frac{1}{4}EC$ (Art. 332.); and in the triangle AEC, $EC : AE :: \sin. E : \cos. I$; therefore, $EC = \frac{\sin. E \times AE}{\cos. I}$; and, by substituting for AE it's value $\frac{\sin. E \times AP}{\cos. I}$ (Art. 347.), we have $CE = \frac{\overline{\sin. E}|^{2} \times AP}{\overline{\cos. I}|^{2}}$; and $\frac{1}{4}EC = \frac{\overline{\sin. E}|^{2} \times AP}{4\,\overline{\cos. I}|^{2}} = \frac{\overline{\sin. E}|^{2} \times V^{2}}{\overline{\cos. I}|^{2} \times 4m}$, the greatest height required.

(352.) Cor. The greatest height varies as $\frac{\overline{\sin. E}|^{2} \times V^{2}}{\overline{\cos. I}|^{2}}$.

SCHOLIUM.

(353.) The theory of the motion of projectiles, given in this section, depends upon three suppositions, which are all inaccurate; 1st. that the force of gravity, in every point of the curve described, is the same; 2d. that it acts in parallel lines; 3d. that the motion is performed in a non-resisting medium. The two former of these, indeed, differ insensibly from the truth. The force of gravity, without the Earth's surface, varies inversely as the square of the distance from the center; and the altitude to which we can project a body from the surface is so small, that the variation of the force, arising from the alteration of the distance from the center of the Earth, may safely be neglected. The direction of the force is every where perpendicular to the horizon; and if perpendiculars be thus drawn, from any two points in the curve which we can cause a body to describe, they may be considered as parallel, since they only meet at, or nearly at, the center of the Earth. Even the resistance of the air does not materially affect the motions of heavy bodies, when they are projected with small velocities. In other cases, however, this resistance is so great as to render the conclusions, drawn from the theory, almost entirely inapplicable in practice. From experiments made to determine the motions of cannon-

balls, it appears that when the initial velocity is considerable, the air's resistance is 20 or 30 times as great as the weight of the ball; and that the horizontal range is often not $\frac{1}{20}$ part of that which the preceding theory leads us to expect. It appears also, that when the angle of elevation is given, the horizontal range varies nearly as the square root of the velocity of projection; and the time of flight as the range; whereas, according to the theory, the time varies as the velocity; and the range as the square of the velocity of projection (Arts. 340. 334.) These experiments, made with great care, and by men of eminent abilities, shew how little the parabolic theory is to be depended upon in determining the motions of military projectiles. See ROBIN'S *New Theory of Gunnery*, and HUTTON'S *Mathematical Dictionary*, article *Gunnery*.

Besides diminishing the velocity of the projectile, the air's resistance will also change it's direction, whenever the body has a rotatory motion about an axis which does not coincide with the direction in which it is moving. For the velocity with which that side of the body, strikes the air, on which the rotatory and progressive motions conspire, is greater than the velocity with which the other side strikes it, where they are contrary to each other; and therefore the resistance of the air, which increases with the velocity, will be greater in the former case than in the latter, and cause the body to deviate from the line of it's motion; this deviation will also be from the *plane* of the first motion, unless the axis of rotation be perpendicular to that plane.

Upon this principle Sir I. NEWTON explains the irregular motion of a tennis-ball*, and the same cause has been assigned by Mr. ROBINS for the deviation of a bullet from the vertical plane†. Mr. EULER, indeed, in his remarks on the *New Theory of Gunnery*, contends that the resistance of the air can neither be increased nor diminished by the rotation of the ball; because such a motion can produce no effect but in the direction of a tangent to the surface of the revolving body; and the tangential force, he affirms, is almost entirely lost. In this instance, the learned writer seems to have been misled by the common theory of resistances, according to which the tangential force produces no effect; whereas, from experiments lately made, with a view to ascertain the quantity and laws of the air's resistance, it appears that every theory which neglects the tangential force must be erroneous.

* Phil. Trans. Vol. VI. p. 3078. MACLAURIN'S *Newton*, p. 120.

† Tracts, Vol. I. pp. 151. 198. 214.

APPENDIX.

ON THE EFFECTS PRODUCED BY WEIGHTS ACTING UPON MACHINES IN MOTION, AND ON THE ROTATION OF BODIES.

THE investigation of the effects produced by bodies when the machines on which they act are in motion, has not usually been introduced into elementary Treatises; but as the theory depends upon the principles already laid down, and may, by the help of the simplest analytical operations, be easily deduced from them, it may not improperly be added, by way of Appendix, here.

PROP. LXXXVIII.

(354.) *To find what weight* x, *placed at* A *upon a machine in motion, resists the rotation as much as* y *placed at* B.

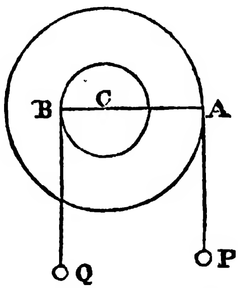
Let a and b be the velocities of the weights; then xa and yb are their momenta; and since these momenta produce equal effects on the machine, or,

are sufficient to balance each other, $xa : yb :: b : a$ (Art. 149.); therefore $xa^2 = yb^2$, and $x = \frac{yb^2}{a^2}$.

Prop. LXXXIX.

(355.) *If two weights acting upon a wheel and axle put the machine in motion, to determine the velocity acquired by the descending body, and the tension of the string by which it acts.*

Let C be the center of motion; CA, CB the radii of the wheel and axle; p and q the two weights, of which p descends; $CA = a$, $CB = b$, then a and b are proportional to the velocities of p and q. And



let $z =$ the weight which q would sustain at p; and $x =$ the weight which placed at p would resist the communication of rotation as much as q resists it; $v =$ the velocity generated in the time t; $m = 16\frac{1}{12}$ feet.

Then $a : b :: q : z = \frac{qb}{a}$, and $x = \frac{qb^2}{a^2}$ (Art. 354.);

hence, $p \quad \frac{qb}{a} =$ the force at p to move the machine, and $p + \frac{qb^2}{a^2} =$ the inertia to be moved,

neglecting the inertia of the machine; consequently,

$$\frac{p - \frac{qb}{a}}{p + \frac{qb^2}{a^2}} = \text{the accelerating force, that of gravity}$$

being represented by unity (Art. 268.); and since $v = 2mft$ (Prop. LXI. Cor. 7.), we have, in this

$$\text{case, } v = \frac{p - \frac{qb}{a}}{p + \frac{qb^2}{a^2}} \times 2mt = \frac{pa^2 - qab}{pa^2 + qb^2} \times 2mt.$$

Again, since $\frac{pa^2 - qab}{pa^2 + qb^2}$ is the accelerating force at p, the moving force, which generates p's velocity, is $\frac{pa^2 - qab}{pa^2 + qb^2} \times p$; therefore $p - \frac{pa^2 - qab}{pa^2 + qb^2} \times p$ is that part of p's weight * which is sustained, or the weight which stretches the string; that is, $\frac{pqb^2 + pqab}{pa^2 + qb^2}$, or $\frac{\overline{a+b}.bpq}{pa^2 + qb^2}$ is the weight which stretches the string AP.

(356.) Cor. 1. The tension of the string AP is just sufficient to sustain the tension of the string BQ; therefore $b : a :: \frac{\overline{a+b}.bpq}{pa^2 + qb^2} : \frac{\overline{a+b}.apq}{pa^2 + qb^2}$ = the tension of the string BQ.

(357.) Cor. 2. The pressure on the center of motion is the sum of the tensions of the strings AP

* In this operation, the moving force and the quantity of matter are, respectively, represented by the weight.

and BQ (Art. 101.), or, $\frac{\overline{a+b}.b}{pa^2+qb^2} \times pq + \frac{\overline{a+b}.a}{pa^2+pb^2} \times pq = \frac{\overline{a+b}|^2.pq}{pa^2+qb^2}$.

(358.) Cor. 3. When a and b are equal, the pressure on the center is $\frac{4pq}{p+q}$.

(359.) Cor. 4 Since s, the space which p descends from rest in t seconds $= mft^2$ (Prop. LXI. Cor. 7.), $s = \frac{pa^2 - qab}{pa^2 + qb^2} \times mt^2$.

(360.) Cor. 5. The same reasoning may be applied when the bodies act upon any other machine.

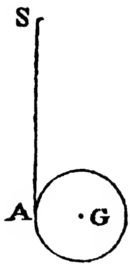
(361.) Cor. 6. If the inertia of the machine is to be taken into consideration, let r be the weight determined by experiment, or calculation, which when placed at p, would resist the communication of rotation as much as the whole machine resists it; then $p - \frac{qb}{a}$ is the moving force at p, and $r + p + \frac{qb^2}{a^2}$ is the quantity of matter to be moved; therefore, $\frac{pa^2 - qab}{ra^2 + pa^2 + qb^2}$ is the accelerating force at p, the accelerating force of gravity being represented by unity.

Prop. XC.

(362.) *If a string be wrapped round a hollow cylinder* G, *and one end fixed at* S, *to find the tension of the string when the cylinder is suffered to descend.*

Let a = the weight of the cylinder, collected in the circumference; x = the tension of the string.

Then, since the motion of the center of gravity of the cylinder is the same at whatever point of the



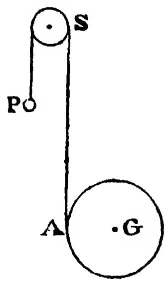
body the force is applied (Art. 182.), $a-x$ is the moving force by which the center of gravity of the cylinder descends, and $\frac{a-x}{a}$ is the accelerating force. Again, x is the weight, or moving force, which applied at the circumference of the cylinder, generates the rotation, and $\frac{x}{a}$ is the accelerating force; and since accelerating forces are proportional to the velocities generated in the same time, and, from the nature of the case, the center of gravity of the cylinder descends as fast as the string is unfolded, that is, the velocities of the center of gravity and rotation are always equal, we have $\frac{a-x}{a}=\frac{x}{a}$; hence, $x=\frac{a}{2}$; or the tension of the string is half the weight of the cylinder.

(363.) Cor. The accelerating force is $\frac{a-x}{a}=\frac{1}{2}$, the accelerating force of gravity being represented by unity.

Prop. XCI.

To find the tension when the string passes over a fixed pulley and a weight is attached to it.

(364.) Let p be the weight of the body attached to the string; x the tension; a the weight of the



cylinder. Then $p - x$ is the moving force on p; $\frac{p-x}{p}$, the accelerating force; $\frac{a-x}{a}$, the force which accelerates the cylinder; $\frac{x}{a}$, the accelerating force which produces the rotation; which quantities are proportional to the velocities generated in the same time. Also, the spaces descended by p and A are, together, always equal to the length of the string disengaged, therefore, $\frac{p-x}{p}+\frac{a-x}{a}=\frac{x}{a}$; hence, $x = \frac{2ap}{2p+a}$.

(365.) Cor. 1. The pressure on the center of the pulley is $2x$, or $\frac{4ap}{2p+a}$.

(366.) Cor. 2. The accelerating force on $p = \frac{p - x}{p} = \frac{2p - a}{2p + a}$.

(367.) Cor. 3. The accelerating force on the cylinder $= \frac{a - x}{a} = \frac{a}{2p + a}$

(368.) Cor. 4. If $p = a$, the body and the cylinder are equally accelerated; that is, they descend at the same rate.

(369.) Cor. 5. If $p = \frac{a}{2}$, the accelerating force on p vanishes, and p remains at rest.

(370.) Cor. 6. If the cylinder be solid and of uniform density, it will appear, nearly in the same manner, that the tension of the string is $\frac{2ap}{3p+a}$; the force which accelerates the cylinder, $\frac{p+a}{3p+a}$; and the force which accelerates p, $\frac{3p - a}{3p + a}$.

Prop. XCII.

(371.) *To find the tension of the string, when the weight of the pulley is taken into the account.*

Let c be the weight which, placed at the circumference of the pulley, would resist the communication of motion as much as the pulley; and let $y =$ the tension of the string SP, $x =$ the tension of SA. Then $\frac{p-y}{p} =$ the accelerating force on p, $\frac{a-x}{a} =$ the

accelerating force on A; $\frac{y-x}{c}$ = the accelerating force on the circumference of the pulley; and $\frac{x}{a}$ = the accelerating force which produces the rotation of the cylinder. Then, as in the last Proposition, $\frac{p-y}{p} + \frac{a-x}{a} = \frac{x}{a}$; also, because the circumference of the pulley always moves as fast as p, $\frac{p-y}{p} = \frac{y-x}{c}$; from which equations we find $x = \frac{2ap + ac}{2p + a + 2c}$; and $y = \frac{\overline{a+c}.2p}{2p + a + 2c}$

(372.) Cor. 1. The force which accelerates the cylinder is $\frac{a-x}{a} = \frac{a+c}{2p + a + 2c}$; and the force which accelerates p, $= \frac{p-y}{p} = \frac{2p - a}{2p + a + 2c}$.

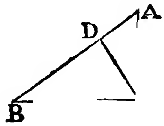
(373.) Cor. 2. The accelerating force being known, the space, time, and velocity, may be found in terms of each other (Prop. LXI. Cor. 7).

Prop. XCIII.

(374.) *If any weights* A, B, C, *act upon a machine and put it in motion, and* x, y, z, *be the spaces described in the direction of gravity,* a, b, c, *the actual velocities of the weights,* $m = 16\frac{1}{12}$ *feet, then* $4m \times \overline{Ax + By + Cz} = Aa^2 + Bb^2 + Cc^2$.

Let AB be the direction of A's motion, AC per-

pendicular to the horizon; take $AB = s$, the space described by A in a very small time; draw BC



parallel to the horizon, and CD perpendicular to AB; let f be the force which accelerates A's motion; F the accelerating force in the direction of gravity.

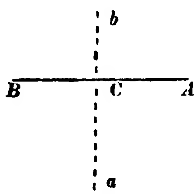
Then $2mf\dot{s} = a\dot{a}$ (VINCE's *Flux.* Art. 82.); and $F : f :: AC : AD :: AB : AC :: \dot{s} : \dot{x}$; therefore, $f\dot{s} = F\dot{x}$, and $2mF\dot{x} = 2mf\dot{s} = a\dot{a}$, consequently $F = \frac{a\dot{a}}{2m\dot{x}}$; hence, the effective moving force on A, in the direction of gravity, $= \frac{Aa\dot{a}}{2m\dot{x}}$, and $A - \frac{Aa\dot{a}}{2m\dot{x}}$ is that part of A's whole weight, or moving force, which is sustained by the action of the other bodies in the system, that is, with which A urges the machine in the direction of gravity. In the same manner, $B - \frac{Bb\dot{b}}{2m\dot{y}}$ is the part of B's moving force sustained, and the weight at A which would balance this : $B - \frac{Bb\dot{b}}{2m\dot{y}} :: \dot{y} : \dot{x}$ (Art. 149.), therefore $\frac{B\dot{y}}{\dot{x}} - \frac{Bb\dot{b}}{2m\dot{x}}$ is the weight at A which would balance B's pressure upon the machine, and $\frac{Bb\dot{b}}{2m\dot{x}} - \frac{B\dot{y}}{\dot{x}}$ is that part of A's moving force which is sustained by B. In the same manner $\frac{Cc\dot{c}}{2m\dot{x}} - \frac{C\dot{z}}{\dot{x}}$ is

that part of A's moving force which is sustained by C; consequently $A-\frac{Aa\dot{a}}{2m\dot{x}}=\frac{Bb\dot{b}}{2m\dot{x}}-\frac{B\dot{y}}{\dot{x}}+\frac{Cc\dot{c}}{2m\dot{x}}-\frac{C\dot{z}}{\dot{x}}$, and $2m\times\overline{A\dot{x}+B\dot{y}+C\dot{z}}=Aa\dot{a}+Bb\dot{b}+Cc\dot{c}$, and taking the fluents, which require no correction, $2m\times\overline{Ax+By+Cz}=\frac{Aa^2}{2}+\frac{Bb^2}{2}+\frac{Cc^2}{2}$, or $4m\times\overline{Ax+By+Cz}=Aa^2+Bb^2+Cc^2$*.

(375.) Cor. If any of the bodies move in a direction opposite to that which is here supposed to be positive, the space described must be reckoned negative.

Ex. 1. If the weights A and B be attached to the lever AB, to find the velocity acquired by A during the motion of the lever, round the pivot C, from an horizontal to a vertical position.

Let $CA=a$, $CB=b$, $v=$ the velocity acquired by A; then $a:b::v:\frac{bv}{a}=$ the velocity acquired



by B; therefore, by the Proposition, $4mAa-4mBb=Av^2+\frac{Bb^2v^2}{a^2}$, and $v^2=4ma^2\times\frac{Aa-Bb}{Aa^2+Bb^2}$;

* For this very concise demonstration, the Author is indebted to the suggestions of the Rev. D. M. Peacock.

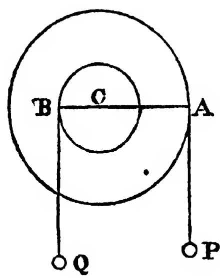
and by extracting the square root of this quantity, v is obtained.

Ex. 2. If the weight p raise q by the wheel and axle, and descend through s feet, to find the velocity acquired by p.

Let $CA = a$, $CB = b$, and v = the velocity required.

$\therefore$ Then $a : b :: v : \frac{bv}{a}$ = q's velocity,

and $a : b :: s : \frac{bs}{a}$ = the space through which

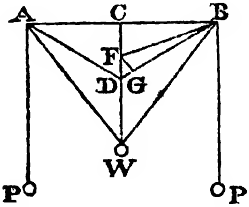


q is raised; therefore, $4mps - 4m \times \frac{bqs}{a} = pv^2 + q \times \frac{b^2v^2}{a^2}$, and $v^2 = 4ms \times \frac{pa^2 - qab}{pa^2 + qb^2}$; whence v is known.

Ex. 3. If two equal weights p, p, attached to a string which passes over the pullies A and B, raise the weight w through the space WD, to find the velocity communicated to W.

Suppose AB to be parallel to the horizon, and WDC perpendicular to it. Take $BC = CA = a$;

$WC = b$; $WB = c$; $DC = x$; $v =$ the velocity of



w; $y =$ the velocity of p; DF the small increment of WD; draw FG at right angles to DB.

Then $v : y :: DF : DG :: DB : DC$, and $v^2 : y^2 :: DB^2 : DC^2 :: a^2 + x^2 : x^2$, therefore $y^2 = \frac{x^2 v^2}{a^2 + x^2}$. Also, WD, or $b - x$ is the space which w has described in a direction perpendicular to the horizon, and $WB - DB$, or $c - \sqrt{a^2 + x^2}$, is the space through which p has descended; therefore, by the Proposition, $4m . 2p . \overline{c - \sqrt{a^2 + x^2}} - 4mw . \overline{b - x} = \frac{2px^2v^2}{a^2 + x^2} + wv^2$, and $v^2 = 4m . \overline{a^2 + x^2} \times \frac{2pc - 2p . \sqrt{a^2 + x^2} - w . \overline{b - x}}{2px^2 + w . \overline{a^2 + x^2}}$; whence v is known.

THE END.